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Classification and partial ordering of reductive Howe dual pairs of classical Lie groups

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Abstract

Using a general method [C. Moeglin, M.-F. Vignéras, J.-L. Waldspurger, Correspondances de Howe sur un Corps p-adique, Lecture Notes in Mathematics, Vol. 1291, Springer, Berlin, 1987] we derive a complete list of conjugacy classes of reductive Howe dual pairs of groups of isometries of real, complex, and quaternionic Hermitian spaces. Moreover, we establish the natural partial ordering on the set of reductive Howe dual pairs which is defined by inclusion modulo conjugacy. As an application, we determine the singularity structure of the orbit space of a pure $SU(n)$ gauge theory over space–time S^4. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The notion of a reductive dual pair of subgroups of a symplectic group has been introduced in the late 1970s by Howe [5] in order to establish a duality relation (which is now called *Howe correspondence*) between representations of different classical Lie groups. It was this relation, rather than the reductive dual pairs themselves, which has attracted a lot of interest and found many applications. Since Howe correspondence is beyond the scope of this paper, for the reader interested in details we give a few references. There are, at first, Howe's articles [5–7] which develop the relevant ideas very clearly. Then in

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[3] some examples are discussed explicitly. More detailed expositions one may find, for instance, in [12,13,16] (though [12] actually addresses the case of p-adic groups). Applications to problems in theoretical physics can be found, for example, in [6,11]. Both these papers, as well as [16], provide, in addition, excellent reference resources for further reading.

Our interest in reductive Howe dual pairs, on the other hand, originates from gauge theory. Let us consider a pure gauge theory, defined on a principal bundle over a compact space–time, with structure group G. The physical degrees of freedom of the theory are contained in the orbit space $\mathcal{M}$ of the action of the gauge group on the space of gauge potentials. So in order to get a deeper insight into the theory, and especially into its quantization, it is necessary to analyze the topological and geometrical structure of $\mathcal{M}$. For non-Abelian G it is clear that $\mathcal{M}$, in general, will not be a smooth manifold. However, as was shown in [8,9], $\mathcal{M}$ is a stratified manifold, i.e. a manifold with singularities which themselves are smooth manifolds again. Moreover, the information about which singularities may occur and how they are patched together is encoded in the partially ordered set of orbit types of the gauge group action (or some derived action, see Section 9). Now, the determination of this set, which may be viewed as a first step towards a detailed study of the structure of $\mathcal{M}$, presupposes knowledge of the reductive Howe dual pairs of the structure group G.

To our knowledge, the classification of reductive Howe dual pairs has been treated explicitly in the literature only for:

(a) symplectic groups, as a special case of groups of isometries of Hermitian spaces. Here one uses tensor product decompositions of the symplectic form (see, for instance, [5,12,13]) and

(b) complex semisimple Lie algebras, using the calculus of roots (see the comprehensive article [14]).

Both in setup (a) and (b) there have been obtained only partial results on the natural partial ordering of reductive Howe dual pairs (see [10] for (a) and [14] for (b)). So in the present paper we aim to give, in a setup similar to (a), a detailed and self-contained exposition of the theory of reductive Howe dual pairs of groups of isometries of real, complex, and quaternionic Hermitian spaces (these groups are listed in Table 1), primarily addressed to the non-specialist. The method we use is taken from [12], Chapter I. We only slightly reformulate it in order to avoid involved tensor products.

The paper is organized as follows: In Section 2 we give the basic definitions and introduce the notion of an irreducible Howe dual pair. As it comes out there are two types of irreducibility. In Section 3 we discuss, as a prerequisite, the case of general linear algebras. Type 1 and type 2 pairs are then classified in Sections 4 and 5, respectively. The results are displayed in Table 4. Section 6 establishes the partial ordering of reductive Howe dual pairs. In Section 7 we discuss some simple examples in detail. As a minor remark, in Section 8 we note that knowledge of the partial ordering provides, in particular, a classification of Kudla's seesaw pairs [10]. Finally, in Section 9 we discuss, by the example of $\mathrm{SU}(n)$, how one can use the results obtained to determine the singularity structure of the orbit space of a pure gauge theory over space–time S^4.

2. Basic definitions

2.1. Reductive Howe dual pairs

Let G denote a group. A *Howe dual pair* in G is an ordered pair of subgroups (H_1, H_2) obeying

$$C_G(H_1) = H_2, \quad C_G(H_2) = H_1.$$

Here C_G means the centralizer in G. The constituents H_1 and H_2 are called *Howe subgroups*. Equivalently, a Howe subgroup is characterized by the property

$$C_G(C_G(H)) = H.$$

The identification of H with the pair $(H, C_G(H))$ yields a 1:1-relation between Howe subgroups and Howe dual pairs.

Any group G possesses the trivial dual pair $(C(G), G)$. A non-trivial pair is, for example, $(\mathrm{SO}(2), \mathrm{SO}(2))$ in the real orthogonal group $\mathrm{O}(2)$.

Let (H_1, H_2) and (D_1, D_2) be Howe dual pairs in G. Clearly, if H_1 and D_1 are conjugate in G then so are H_2 and D_2. Hence conjugacy defines an equivalence relation in the set of Howe dual pairs of G. Now assume that G is a linear Lie group acting on a vector space V. Then a Howe dual pair (H_1, H_2) is called *reductive* iff the induced representations of both H_1 and H_2 on V are completely reducible. Let $\mathcal{H}(G)$ denote the set of conjugacy classes of reductive Howe dual pairs of G. $\mathcal{H}(G)$ carries a natural partial ordering: conjugacy classes $\alpha, \beta \in \mathcal{H}(G)$ obey $\alpha \leq \beta$ iff there are representatives (H_1, H_2) of α and (D_1, D_2) of β such that $H_1 \subseteq D_1$ (then $H_2 \supseteq D_2$).

The notion of reductive Howe dual pair as well as the relations of equivalence and partial ordering extend in an obvious way to algebras.

2.2. Hermitian vector spaces

Let $\mathbb{K}$ be an involutive field. Denote the involution by κ and the center of $\mathbb{K}$ by $\mathbb{K}'$. We restrict our attention to $\mathbb{R}$ (real numbers with identical involution), $\mathbb{C}_1$ and $\mathbb{C}_c$ (complex numbers with identical involution and conjugation, respectively), and $\mathbb{H}$ (quaternions with conjugation). A *Hermitian metric* of dimension n over $\mathbb{K}$ is a matrix $I \in \mathrm{GL}(n, \mathbb{K})$ for which there exists $\varepsilon \in \mathbb{K}'$ such that

$$I^{\dagger} = \varepsilon I.$$

Here $\dagger$ means transposition of matrix and conjugation of entries by κ. The factor ε will be referred to as *flip factor* of I. It obeys

$$\kappa(\varepsilon)\varepsilon = 1.$$

Hermitian metrics I, J over $\mathbb{K}$ are *isometric* iff (i) they have the same dimension n and (ii) there exists $T \in \mathrm{GL}(n, \mathbb{K})$ such that

$$J = T^{\dagger} I T.$$

They are *similar* iff there exists $T \in \mathrm{GL}(n, \mathbb{K})$ and $\beta \in \mathbb{K}'$ such that

$$J = \beta T^{\dagger} I T.$$

Any n-dimensional Hermitian metric I over $\mathbb{K}$ defines an involution $A \mapsto A^I$ on the associative algebra $\mathrm{gl}(n, \mathbb{K})$ by

$$A^I := I^{-1} A^{\dagger} I. \tag{1}$$

By means of this involution the *unitary group* of I is defined as

$$\mathrm{U}_{\mathbb{K}}(I) := \{A \in \mathrm{gl}(n, \mathbb{K}) : A^I A = \mathbf{1}\}.$$

One sees that $\mathrm{U}_{\mathbb{K}}(I)$ consists exactly of the self-isometries of I. Moreover, Hermitian metrics (over one and the same involutive field) are similar iff their unitary groups are isomorphic.

We remark that there is a 1:1-relation between n-dimensional Hermitian metrics I over $\mathbb{K}$ and Hermitian forms $\tilde{I}$ on the right $\mathbb{K}$-vector space $\mathbb{K}^n$. It is given by

$$\tilde{I}(x, y) = \sum_{j,k=1}^{n} \kappa(x_j) I_{jk} y_k \quad \forall\, x, y \in \mathbb{K}^n. \tag{2}$$

The notions of isometry and similarity of Hermitian metrics originate, of course, from the geometric ones defined for Hermitian forms. We shall refer to the pair $(\mathbb{K}^n, I)$ as a *Hermitian space* over $\mathbb{K}$. Finally, a Hermitian subspace of $(\mathbb{K}^n, I)$ is a subspace V of $\mathbb{K}^n$ for which the restriction $\tilde{I}|_V$ is non-degenerate.

In order to classify Hermitian metrics up to isometry (resp. similarity) one occasionally needs, besides dimension n and flip factor ε, the signature s (resp. its modulus) as a third invariant. Recall that it is defined, for metrics which have real eigenvalues, as the number of positive minus the number of negative eigenvalues.

Table 1 lists the isometry and similarity classes of Hermitian metrics over $\mathbb{K} = \mathbb{R}, \mathbb{C}_1, \mathbb{C}_c, \mathbb{H}$, together with the corresponding unitary groups (cf., for instance, [12, Section I.11–I.15]). Note that in case $\mathbb{K} = \mathbb{C}_c$ the flip factor ε is an invariant w.r.t. isometries but not w.r.t. similarity transformations.

We can now formulate the following problem:

Problem. Calculate $\mathcal{H}(\mathrm{U}_{\mathbb{K}}(I))$ for the unitary groups listed in Table 1.

2.3. *Irreducibility*

Assume that we are given a unitary representation of a group G on a Hermitian space $(\mathbb{K}^n, I)$. Since a G-invariant subspace $V \subseteq \mathbb{K}^n$ need not be Hermitian there are two notions of irreducibility: one may require either

(A) *There is no G-invariant Hermitian subspace (irreducibility in the category of Hermitian spaces over $\mathbb{K}$)*, or

Table 1
Real, complex, and quaternionic Hermitian spaces and their unitary groups (the numbers p and q in the last column are defined as $p = \frac{1}{2}(n+s)$ and $q = \frac{1}{2}(n-s)$)

$\mathbb{K}$	Dimension	Isometry classes		Similarity classes		Unitary group
		ε	s	ε	$\lvert s \rvert$	
$\mathbb{R}$	$n \in \mathbb{N}$	$+1$	$n, n-2, \ldots, -n$	$+1$	$n, n-2, \ldots \geq 0$	$\mathrm{O}(p, q)$
	$2n, n \in \mathbb{N}$	-1	–	1	–	$\mathrm{Sp}(n, \mathbb{R})$
$\mathbb{C}_1$	$n \in \mathbb{N}$	$+1$	–	$+1$	–	$\mathrm{O}(n, \mathbb{C})$
	$2n, n \in \mathbb{N}$	-1	–	1	–	$\mathrm{Sp}(n, \mathbb{C})$
$\mathbb{C}_c$	$n \in \mathbb{N}$	$\mathrm{U}(1)$	$n, n-2, \ldots, -n$	–	$n, n-2, \ldots \geq 0$	$\mathrm{U}(p, q)$
$\mathbb{H}$	$n \in \mathbb{N}$	$+1$	$n, n-2, \ldots, -n$	$+1$	$n, n-2, \ldots \geq 0$	$\mathrm{Sp}(p, q)$
	$n \in \mathbb{N}$	-1	–	1	–	$\mathrm{O}^*(n)$

(B) *There is no G-invariant subspace at all (irreducibility in the category of vector spaces over $\mathbb{K}$).*

Obviously, (B) implies (A). Moreover, if I has signature 0 (i.e. if the form defined by I is a scalar product), the conditions are equivalent.

We shall call a unitary representation *irreducible* iff it satisfies condition (A). Evidently, with this definition any finite dimensional unitary representation of G is completely reducible. An irreducible unitary representation of G we shall call *type* 1 iff it satisfies condition (B), and *type* 2 iff not. (This coincides with the terminology of Howe [6].) Finally, one carries over these notions to Howe dual pairs in $\mathrm{U}_{\mathbb{K}}(I)$: Call (H_1, H_2) irreducible (of types 1 and 2) iff the induced unitary representation of the subgroup $H_1 H_2$ of $\mathrm{U}_{\mathbb{K}}(I)$ on $(\mathbb{K}^n, I)$ is irreducible (of corresponding type). Irreducible reductive Howe dual pairs will be abbreviated by IRHDP.

In a similar way one defines irreducible Howe dual pairs in $\mathrm{GL}(n, \mathbb{K})$ and $\mathrm{gl}(n, \mathbb{K})$. Since here the corresponding representations are not unitary one has irreducibility in the usual sense.

The following lemma states that it suffices to classify IRHDP.

Lemma 1. *Let I be a metric of dimension n over $\mathbb{K}$. Let*

$$(\mathbb{K}^n, I) = \bigoplus_{i=1}^{r} (V^i, I^i) \tag{3}$$

be a Hermitian decomposition and let (H_1^i, H_2^i) be IRHDP in $\mathrm{U}_{\mathbb{K}}(I^i)$, $i = 1, \ldots, r$. Then

$$(H_1^1 \times \cdots \times H_1^r, H_2^1 \times \cdots \times H_2^r)$$

is a reductive Howe dual pair in $\mathrm{U}_{\mathbb{K}}(I)$. Conversely, any reductive Howe dual pair of $\mathrm{U}_{\mathbb{K}}(I)$ is of this form.

Proof. Let a Hermitian decomposition (3) be given. Without loss of generality assume $r = 2$ and write operators $T \in \mathrm{gl}(n, \mathbb{K})$ as (2×2)-matrices w.r.t. this decomposition. One only has to check that the centralizer of $H_1^1 \times H_1^2$ is contained in $H_2^1 \times H_2^2$. So assume that $T \in \mathrm{U}_{\mathbb{K}}(I)$ commutes with $H_1^1 \times H_1^2$. Then, for any $A^i \in H_1^i$,

$$\left[\begin{pmatrix} A^1 & 0 \\ 0 & A^2 \end{pmatrix}, \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\right] = \begin{pmatrix} [A^1, T_{11}] & A^1T_{12} - T_{12}A^2 \\ A^2T_{21} - T_{21}A^1 & [A^2, T_{22}] \end{pmatrix} = 0.$$

Since H_1^i always contains $\pm\mathbf{1}_{V^i}$, one may put $A_1 = \mathbf{1}_{V^1}$ and $A_2 = -\mathbf{1}_{V^2}$. It follows that $T_{12} = T_{21} = 0$, and $T \in H_2^1 \times H_2^2$.

Conversely, let (H_1, H_2) be a reductive Howe dual pair in $\mathrm{U}_{\mathbb{K}}(I)$. By complete reducibility of unitary representations, there is a decomposition of $(\mathbb{K}^n, I)$ into a direct orthogonal sum of H_1H_2-irreducible Hermitian subspaces (V^i, I^i). Put $H_j^i := H_j|_{V^i}$, $j = 1, 2$. Then (H_1^i, H_2^i) are IRHDP in $\mathrm{U}_{\mathbb{K}}(I^i)$ and $H_j = H_j^1 \times \cdots \times H_j^r$, $j = 1, 2$. □

As for the equivalence relation, it is clear that reductive Howe dual pairs are conjugate in $\mathrm{U}_{\mathbb{K}}(I)$ iff

(i) the corresponding irreducible orthogonal decompositions of $(\mathbb{K}^n, I)$ are isomorphic,

(ii) the irreducible factors are equivalent in the respective subgroups $\mathrm{U}_{\mathbb{K}}(I^i)$.

The classification of IRHDP of types 1 and 2 will be obtained in different ways. As a prerequisite for both though it is necessary to study the irreducible Howe dual pairs of the algebra $\mathrm{gl}(n, \mathbb{K})$ first.

3. The irreducible Howe dual pairs of $\mathrm{gl}(n, \mathbb{K})$

Before stating the result we shall introduce the notion of $\mathbb{K}$-dual division algebras. Let $\mathbb{L}_1$ be a division algebra over $\mathbb{K}'$ (i.e. an algebra the elements of which are either invertible or zero). As is well known, there are the following possibilities: $\mathbb{L}_1 = \mathbb{R}, \mathbb{C}, \mathbb{H}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, and $\mathbb{L}_1 = \mathbb{C}$ for $\mathbb{K} = \mathbb{C}$. Put

$$\mathbb{L} := \mathbb{L}_1 \cup \mathbb{K}.$$

Since either $\mathbb{L}_1 \subseteq \mathbb{K}$ or $\mathbb{L}_1 \supset \mathbb{K}$, $\mathbb{L}$ is a field. Moreover, $\mathbb{L}$ is the unique simple $(\mathbb{L}_1, \mathbb{K})$-bimodule. Put

$$\mathbb{L}_2 := \mathrm{End}_{(\mathbb{L}_1, \mathbb{K})}(\mathbb{L}). \tag{4}$$

By Schur's lemma, $\mathbb{L}_2$ is also a division algebra over $\mathbb{K}'$. We shall say that $\mathbb{L}_2$ is $\mathbb{K}$-*dual* to $\mathbb{L}_1$. (In order to justify the name 'dual' note that, since simple subalgebras are always Howe [15, Section III.4], $\mathbb{L}_1$ and $\mathbb{L}_2$ can be interchanged in (4).) By definition, $\mathbb{L}_1$ and $\mathbb{L}_2$ have the center in common:

$$\mathbb{L}_1' = \mathbb{L}_1 \cap \mathbb{L}_2 = \mathbb{L}_2'.$$

The values of $\mathbb{L}_2$ and $\mathbb{L}$ for given $\mathbb{L}_1$ are displayed in Table 2.

Here $\mathbb{H}^{\circ}$ denotes the field opposite to $\mathbb{H}$, with multiplication $\alpha \circ \beta := \beta\alpha$. The left action of $\alpha \in \mathbb{H}^{\circ}$ on $\beta \in \mathbb{H}$ is given by $\alpha \circ \beta$.

Table 2
Division algebras $\mathbb{L}_1$ over $\mathbb{K}'$, their $\mathbb{K}$-duals $\mathbb{L}_2$, and their simple $(\mathbb{L}_1, \mathbb{K})$-bimodules $\mathbb{L}$

$\mathbb{K}$	$\mathbb{R}$			$\mathbb{C}$	$\mathbb{H}$	
$\mathbb{L}_1$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{C}$
$\mathbb{L}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}^{op}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{C}$
$\mathbb{L}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$

Theorem 1 [12]. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and let n be a positive integer.*

(a) *Assume that the following data are given:*

 (i) *A division algebra $\mathbb{L}_1$ over $\mathbb{K}'$. Let $\mathbb{L}_2$ denote its $\mathbb{K}$-dual and put $\mathbb{L} := \mathbb{L}_1 \cup \mathbb{K}$.*

 (ii) *Positive integers l_1, l_2 such that*

$$l_1 l_2 \dim_{\mathbb{K}}(\mathbb{L}) = n. \tag{5}$$

Define imbeddings $\phi_i : \mathrm{gl}(l_i, \mathbb{L}_i) \to \mathrm{gl}(l_1 l_2, \mathbb{L})$, $A^i \mapsto \phi_i(A^i)$ by

$$\phi_1(A^1) := \begin{pmatrix} A^1_{11}\mathbf{1}_{n_2} & \cdots & A^1_{1n_1}\mathbf{1}_{n_2} \\ \vdots & \ddots & \vdots \\ A^1_{n_1 1}\mathbf{1}_{n_2} & \cdots & A^1_{n_1 n_1}\mathbf{1}_{n_2} \end{pmatrix},$$

$$\phi_2(A^2) := \begin{pmatrix} A^2 & 0 & \cdots \\ 0 & A^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{6}$$

Then $\phi(\mathrm{gl}(l_1, \mathbb{L}_1))$ and $\phi(\mathrm{gl}(l_2, \mathbb{L}_2))$ constitute an irreducible Howe dual pair in $\mathrm{gl}(n, \mathbb{K})$.

(b) *Any irreducible Howe dual pair of* $\mathrm{gl}(n, \mathbb{K})$ *has this form.*

(c) *Irreducible Howe dual pairs of* $\mathrm{gl}(n, \mathbb{K})$ *are equivalent iff their first (resp. second) constituents are isomorphic.*

Remarks.

1. In (a), the elements of $\phi_i(\mathrm{gl}(l_i, \mathbb{L}_i))$ act as $\mathbb{L}_i$-matrices on $\mathbb{L}^{l_1 l_2}$. By condition (ii), $\mathbb{L}^{l_1 l_2}$ and $\mathbb{K}^n$ are isomorphic over $\mathbb{K}$. So in order to obtain the corresponding $\mathbb{K}$-matrices acting on $\mathbb{K}^n$, i.e. to realize $\phi_i(\mathrm{gl}(l_i, \mathbb{L}_i))$ as subalgebras of $\mathrm{gl}(n, \mathbb{K})$, one has to exploit a particular $\mathbb{K}$-isomorphism $\mathbb{L}^{l_1 l_2} \to \mathbb{K}^n$. Assertion (c) ensures that one may forget about this isomorphism if one is interested in equivalence classes only.
2. The role of ϕ_1 and ϕ_2 is symmetric because it may be interchanged by a $\mathbb{K}$-automorphism of $\mathbb{L}^{l_1 l_2}$ commuting with $\mathbb{L}_1$ and $\mathbb{L}_2$.
3. By (b), any irreducible Howe dual pair of $\mathrm{gl}(n, \mathbb{K})$ is reductive.
4. The theorem traces back to Weyl's double commutant theorem [15, Section III.4]. It applies also to general linear groups if one replaces gl by GL.

Proof. (cf. [12, Section I.18]) (a) Choose a $\mathbb{K}$-isomorphism $\mathbb{L}^{l_1 l_2} \to \mathbb{K}^n$ to identify $\mathrm{gl}(n, \mathbb{K})$ with $\mathrm{End}_{\mathbb{K}}(\mathbb{L}^{l_1 l_2})$. Obviously, $\phi_1(\mathrm{gl}(l_1, \mathbb{L}_1))$ and $\phi_2(\mathrm{gl}(l_2, \mathbb{L}_2))$ commute. In order to show that they centralize each other in $\mathrm{End}_{\mathbb{K}}(\mathbb{L}^{l_1 l_2})$, assume that $T \in \mathrm{End}_{\mathbb{K}}(\mathbb{L}^{l_1 l_2})$ commutes, for example, with $\phi_1(\mathrm{gl}(l_1, \mathbb{L}_1))$. Then $T = \mathrm{diag}(S, \dots, S)$ (l_1 blocks) where S is a $\mathbb{K}$-linear endomorphism of $\mathbb{L}^{l_2}$ commuting with $\mathbb{L}_1$. Hence $S \in \mathrm{gl}(l_2, \mathbb{L}_2)$, and $T \in \phi_2(\mathrm{gl}(l_2, \mathbb{L}_2))$.

(b) Let $(\mathfrak{h}_1, \mathfrak{h}_2)$ be an irreducible Howe dual pair in $\mathrm{gl}(n, \mathbb{K})$. Decompose $\mathbb{K}^n$ into $\mathfrak{h}_2$-invariant subspaces. Since these subspaces are permuted by $\mathfrak{h}_1$, they are all isomorphic and the decomposition is

$$\mathbb{K}^n = W^{l_1} \tag{7}$$

for some $\mathfrak{h}_2$-irreducible subspace W and positive integer l_1. Define

$$\mathbb{L}_1 := C_{\mathrm{End}_{\mathbb{K}}(W)}(\mathfrak{h}_2|_W). \tag{8}$$

By $\mathbb{K}' \subseteq \mathbb{L}_1'$, $\mathbb{L}_1$ is an algebra over $\mathbb{K}'$. By Schur's lemma, it is a division algebra. Denote the $\mathbb{K}$-dual division algebra by $\mathbb{L}_2$ and put $\mathbb{L} := \mathbb{L}_1 \cup \mathbb{K}$. Since $\mathbb{L}$ is the unique simple $(\mathbb{L}_1, \mathbb{K})$-bimodule, W is isomorphic, as such bimodule, to $\mathbb{L}^{l_2}$ for some positive integer l_2. Then $\mathbb{K}^n$ is isomorphic, over $\mathbb{K}$, to $\mathbb{L}^{l_1 l_2}$. Thus we have constructed data (i) and (ii). It remains to check

$$\mathfrak{h}_i = \phi_i(\mathrm{gl}(l_i, \mathbb{L}_i)) \tag{9}$$

for $i = 1, 2$. Since $\mathfrak{h}_2$ is a Howe subalgebra, (8) implies that $\mathfrak{h}_2|_W$ centralizes $\mathbb{L}_1$ in $\mathrm{End}_{\mathbb{K}}(W)$. Then $\mathfrak{h}_2$ centralizes $\phi_1(\mathrm{gl}(l_1, \mathbb{L}_1))$ in $\mathrm{End}_{\mathbb{K}}(W^{l_1})$, which is identified with $\mathrm{gl}(n, \mathbb{K})$. By (a), this yields (9) for $i = 2$ and, in turn, for $i = 1$.

(c) One only has to show that isomorphy implies equivalence. So let $(\mathfrak{h}_1, \mathfrak{h}_2)$ and $(\mathfrak{k}_1, \mathfrak{k}_2)$ be irreducible Howe dual pairs in $\mathrm{gl}(n, \mathbb{K})$ and assume that $\mathfrak{h}_1$ and $\mathfrak{k}_1$ are isomorphic, as algebras over $\mathbb{K}'$. Then they are isomorphic to some $\mathrm{gl}(l_1, \mathbb{L}_1)$, with l_1 and $\mathbb{L}_1$ uniquely determined. By (b), there are $\mathbb{K}$-isomorphisms $\varphi, \psi : \mathbb{L}^{l_1 l_2} \to \mathbb{K}^n$, such that

$$\mathfrak{h}_1 = \varphi \circ \phi_1(\mathrm{gl}(l_1, \mathbb{L}_1)) \circ \varphi^{-1} \quad \text{and} \quad \mathfrak{k}_1 = \psi \circ \phi_1(\mathrm{gl}(l_1, \mathbb{L}_1)) \circ \psi^{-1}.$$

It follows that $\mathfrak{h}_1$ and $\mathfrak{k}_1$ are conjugate by $\psi \circ \varphi^{-1} \in \mathrm{GL}(n, \mathbb{K})$. □

We remark that, equivalently, Howe dual pairs may be constructed using tensor product decompositions

$$\mathbb{K}^n = \mathbb{L}_1^{l_1} \otimes_{\mathbb{M}} \mathbb{L}_2^{l_2},$$

where the field $\mathbb{M}$ depends on $\mathbb{K}$ and $\mathbb{L}_1$. In fact, this is the standard setup used by most authors [5,12,13]. Here the imbeddings $\phi_i : \mathrm{gl}(l_i, \mathbb{L}_i) \to \mathrm{gl}(n, \mathbb{K})$, $A^i \mapsto \phi_i(A^i)$ are defined by

$$\phi_1(A^1)(x \otimes y) := (A^1 x) \otimes y \quad \text{and} \quad \phi_2(A^2)(x \otimes y) := x \otimes (A^2 y).$$

In simple situations, this construction is very obvious. In general, however, we think that the viewpoint we have adopted above (and in what follows) is somewhat easier to handle, especially for explicit calculations.

3.1. Explicit imbeddings

Let $\mathbb{L}_1$ be a division algebra over $\mathbb{K}'$, with $\mathbb{K}$-dual $\mathbb{L}_2$ and simple bimodule $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{K}$, and let l_1, l_2 be positive integers obeying (5). For explicit calculations it is useful to have standard $\mathbb{K}$-isomorphisms $\mathbb{L}^{l_1 l_2} \to \mathbb{K}^n$ at hand. We shall choose them as products of $\mathbb{K}$-isomorphisms $j : \mathbb{L} \to \mathbb{K}^b$, where $b = \dim_{\mathbb{K}} \mathbb{L}$. Such an isomorphism induces an imbedding $\mathrm{gl}(m, \mathbb{L}) \to \mathrm{gl}(bm, \mathbb{K})$, $A \mapsto \widehat{A}$ by requiring

$$\widehat{A} j^m(x) = j^m(A(x)) \tag{10}$$

for any $x \in \mathbb{L}^m$, $A \in \mathrm{gl}(m, \mathbb{L})$.

In case $\mathbb{L} = \mathbb{K}$ we put, of course, $j = \mathrm{id}$. For $\mathbb{L} = \mathbb{C}$, $\mathbb{K} = \mathbb{R}$, put

$$j : \mathbb{C} \to \mathbb{R}^2, x \mapsto (\mathrm{Re}(x), \mathrm{Im}(x)). \tag{11}$$

Then the imbedding $\mathrm{gl}(m, \mathbb{C}) \to \mathrm{gl}(2m, \mathbb{R})$, $A \mapsto \widehat{A}$, is given by replacing the entry A_{ij} by the block

$$\begin{pmatrix} \mathrm{Re}(A_{ij}) & -\mathrm{Im}(A_{ij}) \\ \mathrm{Im}(A_{ij}) & \mathrm{Re}(A_{ij}) \end{pmatrix}. \tag{12}$$

For $\mathbb{L} = \mathbb{H}$, $\mathbb{K} = \mathbb{C}$ write $x \in \mathbb{H}$ as $x^1 + jx^2$, where $x^1, x^2 \in \mathbb{C}$, and put

$$j : \mathbb{H} \to \mathbb{C}^2, x \mapsto (x^1, x^2). \tag{13}$$

The imbedding $\mathrm{gl}(m, \mathbb{H}) \to \mathrm{gl}(2m, \mathbb{C})$, $A \mapsto \widehat{A}$, then replaces A_{ij} by

$$\begin{pmatrix} A^1_{ij} & -\overline{A^2_{ij}} \\ A^2_{ij} & \overline{A^1_{ij}} \end{pmatrix}. \tag{14}$$

Finally, for $\mathbb{L} = \mathbb{H}$, $\mathbb{K} = \mathbb{R}$, take the superposition of the two isomorphisms above. Then

$$j : \mathbb{H} \to \mathbb{R}^4, x \mapsto (x^1, x^2, x^3, -x^4), \tag{15}$$

where $x = x^1 + x^2 i + x^3 j + x^4 k$. Moreover, the imbedding $\mathrm{gl}(m, \mathbb{H}) \to \mathrm{gl}(4m, \mathbb{R})$, $A \mapsto \widehat{A}$, replaces A_{ij} by

$$\begin{pmatrix} A^1_{ij} & -A^2_{ij} & -A^3_{ij} & A^4_{ij} \\ A^2_{ij} & A^1_{ij} & -A^4_{ij} & -A^3_{ij} \\ A^3_{ij} & A^4_{ij} & A^1_{ij} & A^2_{ij} \\ -A^4_{ij} & A^3_{ij} & -A^2_{ij} & A^1_{ij} \end{pmatrix}. \tag{16}$$

As a result, the imbeddings

$$\widehat{\phi_i} : \mathrm{gl}(l_i, \mathbb{L}_i) \xrightarrow{\phi_i} \mathrm{gl}(l_1 l_2, \mathbb{L}) \xrightarrow{\widehat{\ }} \mathrm{gl}(n, \mathbb{K}) \tag{17}$$

assign to $\mathrm{gl}(l_i, \mathbb{L}_i)$ explicit subalgebras of $\mathrm{gl}(n, \mathbb{K})$.

Table 3
Admissible involutive division algebras $\mathbb{L}_1$ over $\mathbb{K}'$ and their $\mathbb{K}$-duals $\mathbb{L}_2$

$\mathbb{K}$	$\mathbb{R}$				$\mathbb{C}_1$	$\mathbb{C}_c$	$\mathbb{H}$	
$\mathbb{L}_1$	$\mathbb{R}$	$\mathbb{C}_1$	$\mathbb{C}_c$	$\mathbb{H}$	$\mathbb{C}_1$	$\mathbb{C}_c$	$\mathbb{R}$	$\mathbb{C}_c$
$\mathbb{L}_2$	$\mathbb{R}$	$\mathbb{C}_1$	$\mathbb{C}_c$	$\mathbb{H}^{\circ}$	$\mathbb{C}_1$	$\mathbb{C}_c$	$\mathbb{H}$	$\mathbb{C}_c$

4. Type 1 irreducible reductive Howe dual pairs

Let $\mathbb{K}$ denote a field with involution κ. To begin with, in analogy to the discussion of $\mathrm{gl}(n, \mathbb{K})$ we shall introduce the notion of $\mathbb{K}$-dual involutive division algebras first. Let $\mathbb{L}_1$ be a division algebra over $\mathbb{K}'$ with involution λ_1. We shall call $\mathbb{L}_1$ *admissible* iff λ_1 and κ coincide on the common subfield $\mathbb{L}_i \cap \mathbb{K}$. Let $\mathbb{L}_2$ be the dual of the underlying division algebra of $\mathbb{L}_1$ w.r.t. the underlying field of $\mathbb{K}$. As one immediately realizes, there is a unique involution λ_2 on $\mathbb{L}_2$ making $\mathbb{L}_2$ admissible and coinciding with λ_1 on the common center $\mathbb{L}_1' = \mathbb{L}_2'$. We shall call $\mathbb{L}_2$, equipped with λ_2, the $\mathbb{K}$-*dual involutive division algebra* of $\mathbb{L}_1$. The admissible involutive division algebras over $\mathbb{K}'$ and their $\mathbb{K}$-duals are listed in Table 3.

The classification result is a natural modification of Theorem 1:

Theorem 2 [12]. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}_1, \mathbb{C}_c, \mathbb{H}$ and let I be a Hermitian metric of dimension n over $\mathbb{K}$. Exclude the case where $\mathbb{K} = \mathbb{H}$, $n = 1$ and I has flip factor -1.*

(a) *Assume that the following data are given:*

(i) *An admissible involutive division algebra $\mathbb{L}_1$ over $\mathbb{K}'$. Let $\mathbb{L}_2$ denote its $\mathbb{K}$-dual and put $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{K}$.*

(ii) *Positive integers l_1, l_2 obeying* (5).

(iii) *Hermitian metrics J_i of dimension l_i over $\mathbb{L}_i$, $i = 1, 2$ such that for both $i = 1, 2$ the following two conditions are satisfied:*

$$\phi_i(A^{J_i}) = \phi_i(A)^I \tag{18}$$

for any $A \in \mathrm{gl}(l_i, \mathbb{L}_i)$ and

$$\mathrm{gl}(l_i, \mathbb{L}_i) = \mathrm{span}_{\mathbb{K}'}(\mathrm{U}_{\mathbb{L}_i}(J_i)). \tag{19}$$

Then $\phi_1(\mathrm{U}_{\mathbb{L}_1}(J_1))$ and $\phi_2(\mathrm{U}_{\mathbb{L}_2}(J_2))$ constitute a type 1 IRHDP of $\mathrm{U}_{\mathbb{K}}(I)$.

(b) *Any type 1 IRHDP of $\mathrm{U}_{\mathbb{K}}(I)$ is of this form.*

(c) *Type 1 IRHDP are equivalent iff they are isomorphic, as ordered pairs of Lie groups.*

Remarks.

1. The formulation of Eq. (18) presupposes that a particular $\mathbb{K}$-isomorphism $\mathbb{L}^{l_1 l_2} \to \mathbb{K}^n$ has been fixed. By (c) one may, as in the case of $\mathrm{gl}(n, \mathbb{K})$, forget about this isomorphism if one is interested in conjugacy classes of Howe dual pairs only.

2. The type 1 IRHDP of the group $O^*(1)$, which are not covered by the theorem, are easily determined in a direct way. One may use, for instance, the complex imaginary unit i as a metric.

Proof. (cf. [12, Section I.18]) Denote $U := U_{\mathbb{K}}(I)$. (a) By (19), $\phi_1(U_{\mathbb{L}_1}(J_1))$ and $\phi_1(\mathrm{gl}(l_1, \mathbb{L}_1))$ have the same centralizer in $\mathrm{gl}(n, \mathbb{K})$. By Theorem 1 this is $\phi_2(\mathrm{gl}(l_2, \mathbb{L}_2))$. Hence the centralizer of $\phi_1(U_{\mathbb{L}_1}(J_1))$ in U is

$$U \cap \phi_2(\mathrm{gl}(l_2, \mathbb{L}_2)).$$

By (18) the intersection is $\phi_2(U_{\mathbb{L}_2}(J_2))$. Similarly, $\phi_1(U_{\mathbb{L}_1}(J_1))$ centralizes $\phi_2(U_{\mathbb{L}_2}(J_2))$ in U so that indeed they constitute a Howe dual pair. Reductivity and type 1 irreducibility are evident.

(b) Let a type 1 IRHDP (H_1, H_2) be given. Define

$$\mathfrak{h}_1 := C_{\mathrm{gl}(n,\mathbb{K})}(H_2), \quad \mathfrak{h}_2 := C_{\mathrm{gl}(n,\mathbb{K})}(\mathfrak{h}_1).$$

One easily verifies

$$H_i = \mathfrak{h}_i \cap U, \quad i = 1, 2. \tag{20}$$

The subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ constitute an irreducible Howe dual pair in $\mathrm{gl}(n, \mathbb{K})$. So Theorem 1 provides division algebras $\mathbb{L}_1$, $\mathbb{L}_2$, dual w.r.t. $\mathbb{K}$ (still without involution), and numbers l_1, l_2 such that

$$\mathfrak{h}_i = \phi_i(\mathrm{gl}(l_i, \mathbb{L}_i)).$$

(Here a particular identification, over $\mathbb{K}$, of $\mathbb{L}^{l_1 l_2}$ and $\mathbb{K}^n$ has been fixed.) Since $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are invariant under the involution induced by I, I defines involutions on $\mathrm{gl}(l_i, \mathbb{L}_i)$, $i = 1, 2$. As a basic fact, these involutions are induced, via (1), by involutions λ_i on $\mathbb{L}_i$ and l_i-dimensional Hermitian metrics J_i over the involutive fields $\mathbb{L}_i$. By construction, J_1 and J_2 satisfy (18). As a consequence,

$$H_i = \mathfrak{h}_i \cap U = \phi_i(U_{\mathbb{L}_i}(J_i)).$$

Next check condition (19): Let Y_i be a complement (over $\mathbb{K}'$) of $\mathrm{span}_{\mathbb{K}'}(H_i)$ in $\mathfrak{h}_i$. By

$$U \cap \mathrm{span}_{\mathbb{K}'}(H_i) \supseteq H_i$$

and (20), U does not intersect with Y_i. On the other hand, U spans $\mathrm{gl}(n, \mathbb{K})$ over $\mathbb{K}'$ (cf. the remark in [12, Section I.14]; for this argument to hold it is necessary that $U \neq O^*(1)$). Hence $Y_i = 0$.

It remains to show that $\mathbb{L}_1$ and $\mathbb{L}_2$ are $\mathbb{K}$-dual: For any $\alpha \in \mathbb{L}_i \cap \mathbb{K}$ one has

$$\kappa(\alpha)\mathbf{1}_n = (\alpha\mathbf{1}_n)^I = \phi_i(\alpha\mathbf{1}_{l_i})^I = \phi_i((\alpha\mathbf{1}_{l_i})^{J_i}) = \phi_i(\lambda_i(\alpha)\mathbf{1}_{l_i}).$$

This shows $\lambda_i(\alpha) \in \mathbb{L}_i \cap \mathbb{K}$, and $\lambda_i(\alpha) = \kappa(\alpha)$, $i = 1, 2$. Moreover, for any $\alpha \in \mathbb{L}_1 \cap \mathbb{L}_2$,

$$\phi_1(\lambda_1(\alpha)\mathbf{1}_{l_1}) = \phi_1((\alpha\mathbf{1}_{l_1})^{J_1}) = \phi_1(\alpha\mathbf{1}_{l_1})^I = \phi_2(\alpha\mathbf{1}_{l_2})^I = \phi_2(\lambda_2(\alpha)\mathbf{1}_{l_2}).$$

Since $\mathbb{L}_1 \cap \mathbb{L}_2$, as the center of $\mathbb{L}_i$, is invariant under λ_i, this implies $\lambda_1(\alpha) = \lambda_2(\alpha)$.

(c) In order to prove assertion (c), we shall proceed in the following way: At first we list, for any given I, the isomorphism types of type 1 IRHDP. Then we shall show that isomorphy implies conjugacy.

4.1. Compatible metrics

Let $\mathbb{L}_1, \mathbb{L}_2$ be $\mathbb{K}$-dual involutive fields and let l_1, l_2 be positive integers subject to condition (5). In order to identify $\mathbb{L}^{l_1 l_2}$ with $\mathbb{K}^n$ we shall use the isomorphisms defined in (11), (13), and (15). The corresponding imbeddings $\mathrm{gl}(l_1 l_2, \mathbb{L}) \to \mathrm{gl}(n, \mathbb{K})$, $A \mapsto \widehat{A}$, are then given by (12), (14), and (16), respectively. These provide imbeddings $\widehat{\phi}_i : \mathrm{gl}(l_i, \mathbb{L}_i) \to \mathrm{gl}(n, \mathbb{K})$ by (17).

Our task is to find the solutions of Eq. (18). In order to do so we shall take arbitrary Hermitian metrics J_1, J_2 and ask for metrics I over $\mathbb{K}$ satisfying this equation. Such metrics we shall call *compatible* with the pair J_1, J_2.

Lemma 2. *Let J_1, J_2 be Hermitian metrics over $\mathbb{L}_1$, $\mathbb{L}_2$, of dimension l_1, l_2 and with flip factor ε_1, ε_2, respectively. Let Δ_n denote the n-dimensional alternating diagonal matrix* $\mathrm{diag}(1, -1, 1, -1, \ldots)$. *Then the Hermitian metrics over $\mathbb{K}$ which are compatible with J_1 and J_2 are given by*

$$\begin{aligned} I_{(\alpha)} &= \Delta_n \widehat{\phi}_1(\alpha J_1)\widehat{\phi}_2(J_2), && \text{if } \mathbb{K} = \mathbb{R}, \mathbb{L}_1 = \mathbb{C}_1, \\ I_{(\alpha)} &= \widehat{\phi}_1(\alpha J_1)\widehat{\phi}_2(J_2), && \text{otherwise,} \end{aligned} \tag{21}$$

where $\alpha \in \mathbb{L}_1'$ such that

$$\alpha^{-1}\lambda_1(\alpha)\varepsilon_1\varepsilon_2 \in \mathbb{K}'. \tag{22}$$

Proof. Let us introduce the notation

$$\begin{aligned} I_0 &:= \Delta_n, && \text{if } \mathbb{K} = \mathbb{R}, \mathbb{L}_1 = \mathbb{C}_1, \\ I_0 &:= \mathbf{1}_n, && \text{otherwise.} \end{aligned}$$

Clearly, I_0 is a Hermitian metric over $\mathbb{K}$. One checks that for any $A \in \mathrm{gl}(l_i, \mathbb{L}_i)$, $i = 1, 2$,

$$\widehat{\phi}_i(A^{\dagger_i}) = \widehat{\phi}_i(A)^{I_0}, \tag{23}$$

where $\dagger_i$ means the canonical involution on $\mathrm{gl}(l_i, \mathbb{L}_i)$, and superscript I_0 means the involution induced by I_0 via (1).

To begin with, assume at first that there is given an $\alpha \in \mathbb{L}_1'$ satisfying (22). Then $I_{(\alpha)}$, defined by (21), is a Hermitian metric over $\mathbb{K}$: To see this, write

$$\begin{aligned} I_{(\alpha)}^{\dagger} &= \widehat{\phi}_1(\alpha J_1)^{\dagger}\widehat{\phi}_2(J_2)^{\dagger} I_0 = I_0 \widehat{\phi}_1(\alpha J_1)^{I_0}\widehat{\phi}_2(J_2)^{I_0} \\ &= I_0 \widehat{\phi}_1(\lambda_1(\alpha) J_1^{\dagger_1})\widehat{\phi}_2(J_2^{\dagger_2}) \\ &= I_0 \widehat{\phi}_1((\alpha^{-1}\lambda_1(\alpha)\varepsilon_1\varepsilon_2)\alpha J_1)\widehat{\phi}_2(J_2). \end{aligned}$$

By (22) the RHS becomes $\alpha^{-1}\lambda_1(\alpha)\varepsilon_1\varepsilon_2 I_{(\alpha)}$.

Next check that $I_{(\alpha)}$ is compatible with J_1, J_2: For any $A \in \mathrm{gl}(l_i, \mathbb{L}_i)$,

$$\begin{aligned}\phi_i(A^{J_i}) &= \phi_i(J_i)^{-1}\phi_i(A^{\dagger_i})\phi_i(J_i) \\ &= \phi_2(J_2)^{-1}\phi_1(\alpha J_1)^{-1}\phi_i(A^{\dagger_i})\phi_1(\alpha J_1)\phi_2(J_2).\end{aligned} \tag{24}$$

Insert $\phi_i(A^{\dagger_i}) = I_0^{-1}\phi_i(A)^{\dagger}I_0$ to obtain $\phi_i(A^{J_i}) = \phi_i(A)^{I_{(\alpha)}}$.

As for the converse assertion, assume that I is a Hermitian metric over $\mathbb{K}$, compatible with J_1, J_2. Then, on the one hand, one has (24) with $\alpha = 1$. On the other hand,

$$\phi_i(A^{J_i}) = \phi_i(A)^I = I^{-1}I_0\phi_i(A)^{I_0}I_0^{-1}I = I^{-1}I_0\phi_i(A^{\dagger_i})I_0^{-1}I.$$

Thus, $I_0^{-1}I\phi_2(J_2)^{-1}\phi_1(J_1)^{-1}$ commutes with $\phi_i(\mathrm{gl}(l_i, \mathbb{K}_i))$ for both $i = 1, 2$. Hence it equals $\phi_1(\alpha \mathbf{1}_{l_1})$ for some $\alpha \in \mathbb{L}_1'$. Then $I = I_{(\alpha)}$. □

Now one may proceed in the following way: for each combination of similarity classes of Hermitian metrics over $\mathbb{L}_1$, $\mathbb{L}_2$ (listed in Table 1) one chooses a pair of representatives J_1, J_2 and determines, by use of (21), the similarity class of $I_{(\alpha)}$, for each admissible value of α. Finally one has to check condition (19). In order to see this procedure working we shall discuss some examples in detail. The complete list of type 1 IRHDP then is contained in Table 4 (See also Table 5.)

In the examples, we shall stick to $\mathbb{K} = \mathbb{R}$. The admissible involutive division algebras $\mathbb{L}_1$ and their duals $\mathbb{L}_2$ can be read off from Table 3.

Example 1. Let us begin with the most simple case $\mathbb{L}_1 = \mathbb{R}$. Then $\mathbb{L}_2 = \mathbb{R}$ and $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{K} = \mathbb{R}$. Hence dimensions obey $l_1 l_2 = n$ and imbeddings $\widehat{\phi}_i$ and ϕ_i coincide. Though any value of α satisfies (22), running α does not change the similarity class of $I_{(\alpha)}$. So one may put $\alpha = 1$. Moreover, condition (19) is satisfied for any real Hermitian metric.

We shall derive relations between the invariants. If J_1, J_2 have flip factor ε_1, ε_2 then $I_{(1)}$ has flip factor $\varepsilon = \varepsilon_1\varepsilon_2$. There are three combinations possible: In case $\varepsilon_1 = \varepsilon_2 = 1$, both J_1 and J_2 have a signature, say s_1 and s_2. Then $I_{(1)}$ has signature $s = s_1 s_2$. With the notation

$$\begin{aligned}&p = \tfrac{1}{2}(n+s), \quad q = \tfrac{1}{2}(n-s), \\ &p_i = \tfrac{1}{2}(l_i + s_i), \quad q_i = \tfrac{1}{2}(l_i - s_i), \qquad i = 1, 2,\end{aligned}$$

this yields the IRHDP

$$(\mathrm{O}(p_1, q_1), \mathrm{O}(p_2, q_2)) \text{ in } \mathrm{O}(p, q),$$

where

$$p = p_1p_2 + q_1q_2, \quad q = p_1q_2 + p_2q_1.$$

In case $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, l_2 is even and $\varepsilon = -1$, for any signature of J_1. Hence there is a sequence of IRHDP

$$(\mathrm{O}(p_1, q_1), \mathrm{Sp}(\tfrac{1}{2}l_2, \mathbb{R})) \text{ in } \mathrm{Sp}(\tfrac{1}{2}n, \mathbb{R}), \qquad \text{where } n = (p_1 + q_1)l_2.$$

Table 4
IRHDP of $U_{\mathbb{K}}(I)$

$U_{\mathbb{K}}(I)$	Type	IRHDP	Conditions
$O(p,q)$	1	$O(p_1,q_1), O(p_2,q_2)$	$p = p_1p_2 + q_1q_2$; $q = p_1q_2 + q_1p_2$
		$U(p_1,q_1), U(p_2,q_2)$	$p = 2(p_1p_2 + q_1q_2)$; $q = 2(p_1q_2 + q_1p_2)$
		$Sp(p_1,q_1), Sp(p_2,q_2)$	$p = 4(p_1p_2 + q_1q_2)$; $q = 4(p_1q_2 + q_1p_2)$
		$Sp(n_1,\mathbb{R}), Sp(n_2,\mathbb{R})$	$p = q$; $p = 2n_1n_2$
		$O(n_1,\mathbb{C}), O(n_2,\mathbb{C})$	$p = q$; $p = n_1n_2$; $n_1, n_2 \neq 1$
		$Sp(n_1,\mathbb{C}), Sp(n_2,\mathbb{C})$	$p = q$; $p = 4n_1n_2$
		$O^*(n_1), O^*(n_2)$	$p = q$; $p = 2n_1n_2$; $n_1, n_2 \neq 1$
	2	$GL(n_1,\mathbb{R}), GL(n_2,\mathbb{R})$	$p = q$; $p = n_1n_2$
		$GL(n_1,\mathbb{C}), GL(n_2,\mathbb{C})$	$p = q$; $p = 2n_1n_2$
		$GL(n_1,\mathbb{H}), GL(n_2,\mathbb{H})$	$p = q$; $p = 4n_1n_2$
$Sp(n,\mathbb{R})$	1	$O(p_1,q_1), Sp(n_2,\mathbb{R})$	$n = (p_1 + q_1)n_2$
		$U(p_1,q_1), U(p_2,q_2)$	$n = (p_1 + q_1)(p_2 + q_2)$
		$O(n_1,\mathbb{C}), Sp(n_2,\mathbb{C})$	$n = 2n_1n_2$; $n_1 \neq 1$
		$Sp(p_1,q_1), O^*(n_2)$	$n = 2(p_1 + q_1)n_2$; $n_2 \neq 1$
	2	Same as$O(n,n)$	
$O(n,\mathbb{C})$	1	$O(n_1,\mathbb{C}), O(n_2,\mathbb{C})$	$n = n_1n_2$
		$Sp(n_1,\mathbb{C}), Sp(n_2,\mathbb{C})$	$n = 4n_1n_2$
	2	$GL(n_1,\mathbb{C}), GL(n_2,\mathbb{C})$	$n = 2n_1n_2$
$Sp(n,\mathbb{C})$	1	$O(n_1,\mathbb{C}), Sp(n_2,\mathbb{C})$	$n = n_1n_2$
	2	$GL(n_1,\mathbb{C}), GL(n_2,\mathbb{C})$	$n = n_1n_2$
$U(p,q)$	1	$U(p_1,q_1), U(p_2,q_2)$	$p = p_1p_2 + q_1q_2$; $q = p_1q_2 + q_1p_2$
	2	$GL(n_1,\mathbb{C}), GL(n_2,\mathbb{C})$	$p = q$; $p = n_1n_2$
$Sp(p,q)$	1	$O(p_1,q_1), Sp(p_2,q_2)$	$p = p_1p_2 + q_1q_2$; $q = p_1q_2 + q_1p_2$
		$Sp(n_1,\mathbb{R}), O^*(n_2)$	$p = q$; $p = n_1n_2$
		$U(p_1,q_1), U(p_2,q_2)$	$p = p_1p_2 + q_1q_2$; $q = p_1q_2 + q_1p_2$
	2	$GL(n_1,\mathbb{R}), GL(n_2,\mathbb{H})$	$p = q$; $p = n_1n_2$
		$GL(n_1,\mathbb{C}), GL(n_2,\mathbb{C})$	$p = q$; $p = n_1n_2$
$O^*(n)$	1	$O(p_1,q_1), O^*(n_2)$	$n = (p_1 + q_1)n_2$; $n_2 \neq 1$ if $n \neq 1$
		$Sp(n_1,\mathbb{R}), Sp(p_2,q_2)$	$n = 2n_1(p_2 + q_2)$
		$U(p_1,q_1), U(p_2,q_2)$	$n = (p_1 + q_1)(p_2 + q_2)$
	2	$GL(n_1,\mathbb{R}), GL(n_2,\mathbb{H})$	$n = 2n_1n_2$
		$GL(n_1,\mathbb{C}), GL(n_2,\mathbb{C})$	$n = 2n_1n_2$

[a] NOTE. Keep $p_i \geq q_i$ throughout.

Finally, in case $\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon = 1$ and $I_{(1)}$ possesses a signature. In order to calculate it, choose for J_1, J_2 the usual symplectic matrices. One half of their eigenvalues is i, the other half is $-$i. Then $I_{(1)}$ has eigenvalues 1 and -1, with the same multiplicity. Hence $s = 0$. So this combination gives rise to the IRHDP

$$(\mathrm{Sp}(\tfrac{1}{2}l_1, \mathbb{R}), \mathrm{Sp}(\tfrac{1}{2}l_2, \mathbb{R})) \text{ in } \mathrm{O}(p, p), \quad \text{where } 2p = l_1 l_2.$$

Example 2. Now let us turn to $\mathbb{L}_1 = \mathbb{C}_1$. Here $\mathbb{L}_2 = \mathbb{C}_1$ and $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{K} = \mathbb{C}$. As a consequence, dimensions are related by $n = 2l_1l_2$. Hermitian metrics J_1 and J_2 are classified by their flip factors $\varepsilon_1, \varepsilon_2$, which may take values ± 1. Moreover, (22) imposes no constraint on the factor α.

Again derive relations between the invariants. $I_{(\alpha)}$ has flip factor $\varepsilon = \varepsilon_1\varepsilon_2$. If $\varepsilon_1 = \varepsilon_2 = 1$ then $I_{(\alpha)}$ has flip factor $\varepsilon = 1$. In order to calculate the signature s of $I_{(\alpha)}$, choose, for instance, $J_i = \mathbf{1}_{l_i}$. Then

$$I_{(\alpha)} = \Delta_n \widehat{\phi}_1(\alpha \mathbf{1}_{l_1})$$

is block diagonal, with $l_1 l_2$ blocks

$$\begin{pmatrix} \mathrm{Re}(\alpha) & -\mathrm{Im}(\alpha) \\ -\mathrm{Im}(\alpha) & -\mathrm{Re}(\alpha) \end{pmatrix}.$$

Since each block has eigenvalues $\pm|\alpha|$, one obtains $s = 0$. One sees that also in this example α does not change the similarity class of $I_{(\alpha)}$. (In general, however, it may do.)

It remains to check condition (19): Obviously, $\mathrm{O}(1, \mathbb{C}) = \{1, -1\}$ does not span $\mathrm{gl}(1, \mathbb{C}) = \mathbb{C}$ over $\mathbb{R}$. One convinces oneself that this is the only exception in case $\mathbb{L}_1 = \mathbb{C}_1$. Thus the type 1 IRHDP constructed here is

$$(\mathrm{O}(l_1, \mathbb{C}), \mathrm{O}(l_2, \mathbb{C})) \text{ in } \mathrm{O}(p, p), \text{ where } p = l_1 l_2 \text{ and } l_1 \neq 1,\ l_2 \neq 1.$$

If $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$ then $\varepsilon = -1$ and we obtain the IRHDP

$$(\mathrm{O}(l_1, \mathbb{C}), \mathrm{Sp}(\tfrac{1}{2}l_2, \mathbb{C})) \text{ in } \mathrm{Sp}(\tfrac{1}{2}n, \mathbb{R}), \text{ where } 2l_1 l_2 = n,\ l_1 \neq 1.$$

Finally, if $\varepsilon_1 = \varepsilon_2 = -1$ then $\varepsilon = 1$, and both l_1 and l_2 are even. In order to compute the signature s of $I_{(\alpha)}$, choose for J_1, J_2 the usual symplectic metrics and write

$$I_{(\alpha)} = \Delta_n \widehat{\phi}_1(\alpha \mathbf{1}_{l_1}) \widehat{\phi}_1(J_1) \widehat{\phi}_2(J_2).$$

Since $\Delta_n \widehat{\phi}_1(\alpha \mathbf{1}_{l_1})$ and $\widehat{\phi}_1(J_1)\widehat{\phi}_2(J_2)$ commute, s is the product of their signatures. It follows that $s = 0$. The corresponding IRHDP is

$$(\mathrm{Sp}(\tfrac{1}{2}l_1, \mathbb{C}), \mathrm{Sp}(\tfrac{1}{2}l_2, \mathbb{C})) \text{ in } \mathrm{O}(p, p), \quad \text{where } l_1 l_2 = p.$$

Example 3. As a last example, consider $\mathbb{L}_1 = \mathbb{C}_c$. Here $\mathbb{L}_2 = \mathbb{C}_c$ and $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{K} = \mathbb{C}$. Hence dimensions are subject to $n = 2l_1 l_2$. Moreover, metrics over $\mathbb{C}_c$ are classified up to similarity by their signature, and not by their flip factor, which may take any complex value of modulus 1. So choose decompositions $l_i = p_i + q_i$, $i = 1, 2$, and put

$$J_i = \mathrm{diag}(\mathbf{1}_{p_i}, -\mathbf{1}_{q_i}), \quad i = 1, 2.$$

By (22) the range of α is then restricted by the requirement that $\alpha^{-1}\overline{\alpha}$ be real. Up to a real factor, which does not change the similarity class of $I_{(\alpha)}$, there are two solutions: $\alpha = 1$ and $\alpha = \mathrm{i}$. $I_{(1)}$ has flip factor $\varepsilon = 1$ and signature $s = 2s_1 s_2$, whereas $I_{(i)}$ has flip factor -1 (and therefore no further invariant). The corresponding IRHDP are

$$(\mathrm{U}(p_1, q_1), \mathrm{U}(p_2, q_2)) \text{ in } \mathrm{O}(p, q),$$

where

$$p = 2(p_1 p_2 + q_1 q_2) \quad \text{and} \quad q = 2(p_1 q_2 + q_1 p_2)$$

and

$$(\mathrm{U}(p_1, q_1), \mathrm{U}(p_2, q_2)) \text{ in } \mathrm{Sp}(\tfrac{1}{2}n, \mathbb{R}), \qquad \text{where } 2(p_1 + q_1)(p_2 + q_2) = n,$$

respectively. (Note that here the imbeddings of $\mathrm{U}(p_i, q_i)$ into $\mathrm{gl}(n, \mathbb{R})$ are different, depending on whether they lead to a Howe subgroup in $\mathrm{O}(p, q)$ or $\mathrm{Sp}(m, \mathbb{R})$.)

4.2. Passage to conjugacy classes

In this section we shall prove assertion (c) of Theorem 2, i.e. that isomorphy implies equivalence. So let (H_1, H_2) and (D_1, D_2) be type 1 IRHDP. By assertion (b), there are admissible division algebras $\mathbb{L}_1$ and $\mathbb{M}_1$ with $\mathbb{K}$-duals $\mathbb{L}_2$ and $\mathbb{M}_2$ and simple bimodules $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{K}$ and $\mathbb{M} = \mathbb{M}_1 \cup \mathbb{K}$, metrics J_1, J_2 and K_1, K_2 of dimension l_1, l_2 and m_1, m_2, and $\mathbb{K}$-isomorphisms $\varphi : \mathbb{L}^{l_1 l_2} \to \mathbb{K}^n$ and $\psi : \mathbb{M}^{m_1 m_2} \to \mathbb{K}^n$ such that

$$H_i = \varphi_i(\mathrm{U}_{\mathbb{L}_i}(J_i)) \text{ and } D_i = \psi_i(\mathrm{U}_{\mathbb{M}_i}(K_i)), \quad i = 1, 2,$$

respectively. Here the imbeddings $\varphi_i, \psi_i : \mathrm{gl}(l_i, \mathbb{L}_i) \to \mathrm{gl}(n, \mathbb{K})$ are defined by

$$\varphi_i(A) = \varphi \circ \phi_i(A) \circ \varphi^{-1} \text{ and } \psi_i = \psi \circ \phi_i(A) \circ \psi^{-1}$$

for $A \in \mathrm{gl}(l_i, \mathbb{L}_i)$, $i = 1, 2$. If the pairs are isomorphic, $\mathbb{L}_i = \mathbb{M}_i$, $l_i = m_i$, and J_i and K_i are similar, $i = 1, 2$. In fact, J_i and K_i may be chosen isometric, and by possibly modifying ψ one may even assume $J_i = K_i$, $i = 1, 2$. Then φ_i and ψ_i are two representations of $\mathrm{U}_{\mathbb{L}_i}(J_i)$. Define

$$T := \psi\varphi^{-1}.$$

T intertwines φ_i and ψ_i.

$$T \circ \varphi_i(A) = \psi_i(A) \circ T \quad \forall\, A \in \mathrm{gl}(l_i, \mathbb{L}_i)\,, \; i = 1, 2. \tag{25}$$

Hence $D_i = T H_i T^{-1}$, $i = 1, 2$, i.e. the pairs are conjugate in $\mathrm{GL}(n, \mathbb{K})$. Unfortunately, in general T is not necessarily unitary w.r.t. I: Since φ_i, ψ_i preserve involution, (25) implies

$$T^I T \circ \varphi_i(A) = \varphi_i(A) \circ T^I T \quad \forall\, A \in \mathrm{gl}(l_i, \mathbb{L}_i), \; i = 1, 2.$$

It follows that

$$T^I T = \varphi_1(\beta \mathbf{1}_{l_1})$$

for some $\beta \in \mathbb{L}_1'$. Therefore, in order to obtain conjugacy in $\mathrm{U}_{\mathbb{K}}(I)$, one has to find $S \in \mathrm{gl}(n, \mathbb{K})$ normalizing H_i such that $T \circ S \in \mathrm{U}_{\mathbb{K}}(I)$.

Obviously, $\lambda_1(\beta) = \beta$. From Table 3 one learns that this implies $\beta \in \mathbb{K}'$, except for the case $\mathbb{K} = \mathbb{R}$ and $\mathbb{L}_1 = \mathbb{C}_1$. However, in this case there exists $\gamma \in \mathbb{C}$ such that $\gamma^2 = \beta$. So one may put

$$S := \varphi_1(\gamma^{-1}\mathbf{1}_{l_1}).$$

In any of the other cases one may assume $\beta \in \mathbb{K}'$. Let us investigate which values β may take then. To this end, for given involutive field $\mathbb{M}$ and metric J of dimension m over $\mathbb{M}$ let $W(J)$ denote the set of scalars $\beta \in \mathbb{M}'$ for which there exists $A \in \mathrm{gl}(m, \mathbb{M})$ such that

$$A^J A = \beta\mathbf{1}_m.$$

Determine $W(J)$ for the Hermitian spaces, listed in Table 1, by means of the following simple criterion: An element $\beta \in \mathbb{M}'$ belongs to $W(J)$ iff the metrics J and βJ are isometric. (Because in this case there exists $A \in \mathrm{gl}(m, \mathbb{M})$ such that

$$\beta J = A^* JA = JA^J A.)$$

Then check, using Table 4, that all but two type 1 IRHDP obey

$$W(I) \subseteq \varphi_1(W(J_1))\varphi_2(W(J_2)). \tag{26}$$

The exceptions are

(a) $(\mathrm{U}(p_1, q_1), \mathrm{U}(p_2, q_2))$ in $\mathrm{Sp}(n, \mathbb{R})$

and

(b) $(\mathrm{U}(p_1, q_1), \mathrm{U}(p_2, q_2))$ in $\mathrm{O}^*(n)$,

where $(p_1 + q_1)(p_2 + q_2) = n$ for both pairs.

In case (26) holds one finds operators $S_i \in \mathrm{GL}(l_i, \mathbb{L}_i)$ satisfying

$$S_i^{J_i} S_i = \beta_i \mathbf{1}_{l_i}, \quad i = 1, 2 \quad \text{and} \quad \beta_1\beta_2 = \beta.$$

So one may put

$$S := (\varphi_1(S_1)\varphi_2(S_2))^{-1}.$$

For the exceptions (a) and (b) we shall give an S explicitly. Obviously, it is sufficient that S satisfies

$$S^I S = -\mathbf{1}_n. \tag{27}$$

Choose, in the setup explained in Section 3.1, the following metrics:

$$J_1 = \mathrm{i}\,\mathrm{diag}(\mathbf{1}_{p_1}, -\mathbf{1}_{q_1}), \quad J_2 = \mathrm{diag}(\mathbf{1}_{p_2}, -\mathbf{1}_{q_2}), \quad I = \widehat{\phi}_1(J_1)\widehat{\phi}_2(J_2).$$

Put $S := \Delta_{2n}$ in case (a) and $S := j\mathbf{1}_n$ in case (b). Then S obeys (27), as well as

$$S\widehat{\phi}_1(A)S^{-1} = \widehat{\phi}_1(\overline{A})$$

for any $A \in \mathrm{gl}(p_1 + q_1, \mathbb{C})$. Since $\mathrm{U}(p_1, q_1)$, if defined by J_1 above, is invariant under conjugation $A \mapsto \overline{A}$, S normalizes H_1 and, consequently, also H_2.

This concludes the proof of Theorem 2 and the discussion of type 1 IRHDP. □

5. Type 2 irreducible reductive Howe dual pairs

The occurrence of type 2 IRHDP is restricted to the unitary groups of hyperbolic Hermitian spaces. As it comes out, these pairs are closely related to Lagrangian subspaces. Let us briefly recall the relevant notions:

A Hermitian metric I of dimension n over $\mathbb{K}$ is called *hyperbolic* iff $\mathbb{K}^n$ is the direct sum of two isotropic subspaces. These subspaces are necessarily maximal isotropic and of the same dimension. It follows that a hyperbolic Hermitian space has even dimension. Generally, a maximal isotropic subspace of a hyperbolic Hermitian space is called *Lagrangian*.

Let X be a Lagrangian subspace and let

$$S(X) := \{T \in \mathrm{U}_{\mathbb{K}}(I) : TX = X\} \tag{28}$$

denote its stabilizer in $\mathrm{U}_{\mathbb{K}}(I)$. As a basic fact, restriction to X yields a Lie group isomorphism $S(X) \to \mathrm{GL}(X)$. In particular, any transformation $T' \in \mathrm{GL}(X)$ possesses a unique unitary prolongation $T \in \mathrm{U}_{\mathbb{K}}(I)$.

We shall need the following two special properties of isotropic subspaces:

Lemma 3. *Let I be a Hermitian metric of dimension n over $\mathbb{K}$ and let $X \subseteq \mathbb{K}^n$ be an isotropic subspace. Then*
(a) $X^{\perp\perp} = X$,
(b) *if $X = X^{\perp}$ then I is hyperbolic and X is a Lagrangian subspace.*

Remarks. Orthogonal complements are taken in $\mathbb{K}^n$ and w.r.t. I.

Proof. Choose a basis $\{e_1, \dots, e_m\}$ in X. Then there exist $f_i \in \mathbb{K}^n$, $i = 1, \dots, m$, such that

$$\tilde{I}(e_i, f_j) = \delta_{ij} \text{ and } \tilde{I}(f_i, f_j) = 0\,.$$

Here $\tilde{I}$ denotes the Hermitian form defined by I via (2). Put $Y = \mathrm{span}_{\mathbb{K}}(f_1, \dots, f_m)$. Then Y is isotropic and $X \oplus Y$ is a hyperbolic Hermitian subspace of $\mathbb{K}^n$. Moreover,

$$\mathbb{K}^n = (X \oplus Y) \oplus (X \oplus Y)^{\perp},$$

where the sum is orthogonal w.r.t. I. As a consequence,

$$X^{\perp} = X \oplus (X \oplus Y)^{\perp}. \tag{29}$$

So if $X = X^{\perp}$ then $(X \oplus Y)^{\perp} = 0$. This proves assertion (b). As for (a), (29) implies $X^{\perp\perp} = X^{\perp} \cap (X \oplus Y)^{\perp\perp}$. By $(X \oplus Y)^{\perp\perp} = X \oplus Y$ (as for any Hermitian subspace), the intersection equals X. □

Lemma 4. *Let I be a Hermitian metric of dimension n over $\mathbb{K}$. If $\mathrm{U}_{\mathbb{K}}(I)$ possesses a type 2 IRHDP (H_1, H_2) then I is hyperbolic and there exists a Lagrangian subspace invariant under $H_1 H_2$.*

Proof. By assumption, there is a degenerate $H_1 H_2$-invariant subspace $X_0 \subset \mathbb{K}^n$. Put $X := X_0 \cap X_0^{\perp}$. X is non-trivial, isotropic, and $H_1 H_2$-invariant. Moreover, $X \subseteq X^{\perp}$, where $X^{\perp}$ is also invariant. Since the Howe dual pair (H_1, H_2) is reductive, one finds an invariant subspace $W \subset X^{\perp}$ such that $X^{\perp} = X \oplus W$.

W is non-degenerate: To see this, let $w \in W$. If $w \in W^{\perp}$ then $w \in X^{\perp\perp}$. Lemma 3(a) implies $w \in X$, hence $w = 0$.

Thus, irreducibility of (H_1, H_2) implies $W = \{0\}$ and, consequently, $X = X^{\perp}$. Then, by Lemma 3(b), I is hyperbolic and X is a Lagrangian subspace. □

Theorem 3 [12]. *Let I be a hyperbolic Hermitian metric of dimension n over $\mathbb{K}$.*
(a) *Let $X \subseteq \mathbb{K}^n$ be a Lagrangian subspace and let $(H_{\underline{1}}, H_{\underline{2}})$ be an irreducible Howe dual pair in* GL(X). *Then the unitary prolongations H_i of $H_{\underline{i}}$, $i = 1, 2$, constitute a type 2 IRHDP of* $\mathrm{U}_{\mathbb{K}}(I)$.
(b) *Any type 2 IRHDP of* $\mathrm{U}_{\mathbb{K}}(I)$ *is of this form.*
(c) *Type 2 IRHDP are equivalent iff they are isomorphic.*

Proof. (cf. [12, Section I.18]) Again denote the stabilizer (28) of X in $\mathrm{U}_{\mathbb{K}}(I)$ by $S(X)$.

(a) Given X one finds a complementary Lagrangian subspace $Y \subset \mathbb{K}^n$ with $S(X) = S(Y)$. Write operators $T \in \mathrm{gl}(n, \mathbb{K})$ as (2×2)-block matrices w.r.t. the decomposition $\mathbb{K}^n = X \oplus Y$. Put $A_0 := \mathrm{diag}(2\mathbf{1}_X, \frac{1}{2}\mathbf{1}_Y)$. A_0 is in the center of $S(X)$, hence $A_0 \in H_1 \cap H_2$. Now assume that there is given $T \in \mathrm{U}_{\mathbb{K}}(I)$ commuting with H_1. Then, in particular,

$$[T, A_0] = \begin{pmatrix} 0 & -\frac{3}{2}T_{12} \\ \frac{3}{2}T_{21} & 0 \end{pmatrix} = 0.$$

Thus $T \in S(X)$, and $T|_X \in H_{\underline{2}}$. Since unitary prolongation is unique, $T \in H_2$. So H_2 centralizes H_1 (and vice versa by the same argument).

(b) Let (H_1, H_2) be a type 2 IRHDP of $\mathrm{U}_{\mathbb{K}}(I)$. By Lemma 4 there is a $H_1 H_2$-invariant Lagrangian subspace X. Since H_1 and H_2 are contained in $S(X)$, (H_1, H_2) is a Howe dual pair in $S(X)$. So restriction to X yields a Howe dual pair (obviously irreducible) in GL(X), with unitary prolongation (H_1, H_2).

(c) As usual, one only has to show that isomorphy implies equivalence. As a basic fact, any two Lagrangian subspaces are conjugate w.r.t. the action of $\mathrm{U}_{\mathbb{K}}(I)$. Hence for any two type 2 IRHDP of $\mathrm{U}_{\mathbb{K}}(I)$ one finds equivalent pairs leaving invariant a given Lagrangian subspace X. Now if these pairs are isomorphic then, by Theorem 1, their restrictions to X are equivalent in GL(X). Hence the pairs are equivalent in $S(X)$. □

To complete the classification it suffices to list the hyperbolic ones among the Hermitian spaces over $\mathbb{K} = \mathbb{R}, \mathbb{C}_1, \mathbb{C}_c, \mathbb{H}$. As is well known, these are the ones which have either zero signature, or no signature and even dimension. Their unitary groups are: $\mathrm{O}(n, n)$, $\mathrm{Sp}(n, \mathbb{R})$, $\mathrm{O}(2n, \mathbb{C})$, $\mathrm{Sp}(n, \mathbb{C})$, $\mathrm{U}(n, n)$, $\mathrm{Sp}(n, n)$, $\mathrm{O}^*(2n)$, where n is a positive integer.

This concludes the classification of IRHDP of the classical Lie groups. The results are listed in Table 4.

6. The natural partial ordering of reductive Howe dual pairs

Throughout this section, let I be a Hermitian metric of dimension n over $\mathbb{K}$. In order to establish the natural partial ordering relation on $\mathcal{H}(\mathrm{U}_{\mathbb{K}}(I))$ we shall determine the direct successors of each element.

A remark on terminology: when talking about Howe dual pairs in the following we shall always mean conjugacy classes. Moreover, if H_1, H_2, H_3 are subgroups of $\mathrm{U}_{\mathbb{K}}(I)$ then we shall say that H_2 *separates* H_1 and H_3 iff $H_1 \subset H_2 \subset H_3$ (proper inclusion).

To begin with, we state that it suffices to know the direct successors of IRHDP:

Lemma 5. *Let (H_1, H_2) be a reductive Howe dual pair in $\mathrm{U}_{\mathbb{K}}(I)$. Then those direct successors of (H_1, H_2) which have the same or a greater number of irreducible factors are obtained from (H_1, H_2) by replacing precisely one of the irreducible factors by any one of its direct successors.*

Proof. The reductive Howe dual pairs produced in this way are obviously direct successors of (H_1, H_2). Conversely, let (D_1, D_2) be a direct successor of (H_1, H_2). Choose representatives (denoted by the same letters) s.t. $H_1 \subseteq D_1$. Consider the subgroup L of $\mathrm{U}_{\mathbb{K}}(I)$ generated by H_2 and D_1. Decompose $(\mathbb{K}^n, I) = \bigoplus_{i=1}^{l}(W^i, I^i)$ into L-irreducible Hermitian subspaces. Since this decomposition is coarser than both the H_1H_2-irreducible and the D_1D_2-irreducible one, $(H_1|_{W^i}, H_2|_{W^i})$ and $(D_1|_{W^i}, D_2|_{W^i})$ are Howe dual pairs in $\mathrm{U}_{\mathbb{K}}(I^i)$, and

$$(H_1, H_2) = (H_1|_{W^1} \times \cdots \times H_1|_{W^l}, H_2|_{W^1} \times \cdots \times H_2|_{W^l}),$$
$$(D_1, D_2) = (D_1|_{W^1} \times \cdots \times D_1|_{W^l}, D_2|_{W^1} \times \cdots \times D_2|_{W^l}).$$

Now if $H_1|_{W^i} \neq D_1|_{W^i}$ for more than one index i, say for $i = 1, 2$, then

$$D_1|_{W^1} \times H_1|_{W^2} \times \cdots \times H_1|_{W^l}$$

is a Howe subgroup of $\mathrm{U}_{\mathbb{K}}(I)$ separating H_1 and D_1. Thus $H_1|_{W^i} \neq D_1|_{W^i}$ for precisely one index $i = k$. It is clear that then $(D_1|_{W^k}, D_2|_{W^k})$ has to be a direct successor of $(H_1|_{W^k}, H_1|_{W^k})$ in $\mathrm{U}_{\mathbb{K}}(I^k)$.

It remains to show that $(H_1|_{W^k}, H_2|_{W^k})$ is an *irreducible* factor of (H_1, H_2), i.e. that H_1H_2 acts irreducibly on W^k. In order to see that, consider the subgroup H_1D_2 of $\mathrm{U}_{\mathbb{K}}(I)$ and decompose

$$(W^k, I^k) = \bigoplus_{i=1}^{t}(W^{ki}, I^{ki}) \tag{30}$$

into H_1D_2-irreducible Hermitian subspaces. This decomposition is finer than both the decompositions of (W^k, I^k) into H_1H_2- and into D_1D_2-irreducible subspaces. Now if both H_1H_2 and D_1D_2 would act reducibly on (W^k, I^k) then it was *properly* finer (otherwise (W^k, I^k) was not L-irreducible). Then $H_1|_{W^k}$ and $D_1|_{W^k}$ would be separated by the Howe subgroup $H_1|_{W^{k1}} \times \cdots \times H_1|_{W^{kt}}$ of $\mathrm{U}_{\mathbb{K}}(I^k)$. Thus, at least one of the groups H_1H_2 or D_1D_2

acts irreducibly on (W^k, I^k). By the assumption on the number of irreducible factors, this is $H_1 H_2$. □

Note that if (D_1, D_2) is a direct successor of (H_1, H_2) with less irreducible factors, then (H_2, H_1) is a direct successor of (D_2, D_1) with more irreducible factors, hence meets the assumption of the lemma.

We proceed with the determination of the direct successors of IRHDP. Thereby we shall discuss the following cases separately, in the following order: reducible direct successors, type 2 irreducible direct successors of type 2 IRHDP, type 1 irreducible direct successors of type 1 IRHDP, and type 2 irreducible direct successors of type 1 IRHDP.

6.1. Reducible direct successors of IRHDP

Proposition 1. *Let (H_1, H_2) be an IRHDP of $\mathrm{U}_{\mathbb{K}}(I)$. Assume that*

$$(\mathbb{K}^n, I) = (V^1, I^1) \oplus (V^2, I^2)$$

is an H_1-invariant Hermitian decomposition. Then

$$(D_1, D_2) = (H_1|_{V^1} \times H_1|_{V^2}, C_{\mathrm{U}_{\mathbb{K}}(I^1)}(H_1|_{V^1}) \times C_{\mathrm{U}_{\mathbb{K}}(I^2)}(H_1|_{V^2})) \tag{31}$$

is a reducible direct successor of (H_1, H_2). Conversely, any reducible direct successor of (H_1, H_2) is of this form.

Remarks. We shall say that the direct successor (D_1, D_2) of (H_1, H_2) is obtained by *splitting*, and that the direct successor (H_2, H_1) of (D_2, D_1) is obtained by *inverse splitting*.

Proof. Assume at first that a Hermitian decomposition is given. One straightforwardly checks that (31) is a reductive Howe dual pair in $\mathrm{U}_{\mathbb{K}}(I)$. Clearly, $H_1 \subset D_1$. Assume that (T_1, T_2) is a Howe dual pair obeying $H_1 \subseteq T_1 \subseteq D_1$. Then restriction to V^i of this relation yields $T_1|_{V^i} = H_1|_{V^i}, i = 1, 2$. Hence either $(T_1, T_2) = (H_1, H_2)$ (if (T_1, T_2) is irreducible) or $(T_1, T_2) = (D_1, D_2)$ (if it is reducible).

Conversely, let (T_1, T_2) be a reducible direct successor of (H_1, H_2). Then there is a $T_1 T_2$-invariant (hence, in particular, H_1-invariant) Hermitian decomposition $(\mathbb{K}^n, I) = (V^1, I^1) \oplus (V^2, I^2)$. Let (D_1, D_2) denote the direct successor (31) of (H_1, H_2) defined by this decomposition. By $(T_1, T_2) = (T_1|_{V^1} \times T_1|_{V^2}, T_2|_{V^1} \times T_2|_{V^2})$, D_1 commutes with T_2. Hence $T_1 \subseteq D_1$. It follows $T_1 = D_1$ and, in turn, $T_2 = D_2$. □

Example 4. Consider the IRHDP $(\mathrm{O}(n), \mathrm{O}(m))$ in $\mathrm{O}(nm)$. Here any decomposition is Hermitian. Since the dimension of $\mathrm{O}(n)$-invariant subspaces is a multiple of n, possible decompositions are $\mathbb{R}^{nm} = \mathbb{R}^{nm_1} \oplus \mathbb{R}^{nm_2}$ where $m_1 + m_2 = m$. The corresponding direct successors are

$$(\mathrm{O}(n) \times \mathrm{O}(n), \mathrm{O}(m_1) \times \mathrm{O}(m_2)).$$

6.2. Type 2 irreducible direct successors of type 2 IRHDP

Proposition 2. *Assume that I is hyperbolic and let (H_1, H_2) be a type 2 IRHDP in $\mathrm{U}_{\mathbb{K}}(I)$. Then (H_1, H_2) has the following direct type 2 irreducible successors (D_1, D_2):*

H_1	D_1	*Condition*
$\mathrm{GL}(n_1, \mathbb{R})$	$\mathrm{GL}(n_1, \mathbb{C})$	n_2 even
$\mathrm{GL}(n_1, \mathbb{C})$	$\mathrm{GL}(n_1, \mathbb{H})$	n_2 even
	$\mathrm{GL}(2n_1, \mathbb{R})$	–
$\mathrm{GL}(n_1, \mathbb{H})$	$\mathrm{GL}(2n_1, \mathbb{C})$	–

(32)

Remarks.
1. Since a type 2 IRHDP is uniquely determined by the isomorphism type of each of its constituents, in Table (32) it suffices to list the first one.
2. As imbeddings $\mathrm{GL}(l_1, \mathbb{C}) \subseteq \mathrm{GL}(2l_1, \mathbb{R})$ and $\mathrm{GL}(l_1, \mathbb{H}) \subseteq \mathrm{GL}(2l_1, \mathbb{C})$ one may choose, for instance, (12) and (14), respectively.
3. One sees that in case $\mathbb{K} = \mathbb{C}_1$ or $\mathbb{C}_c$ type 2 IRHDP do not have type 2 irreducible direct successors.
4. The proposition provides, in particular, the irreducible direct successors of irreducible Howe dual pairs in $\mathrm{GL}(m, \mathbb{K})$, where $2m = n$.

Proof. To begin with, assume that (D_1, D_2) is one of the pairs in Table (32). Choose a representative, denoted by the same letters. H_1, as a subgroup of D_1, generates a Howe subgroup S of $\mathrm{U}_{\mathbb{K}}(I)$. S is either a unitary or a general linear group or a product thereof. Since there is no such group separating H_1 and D_1, $S = H_1$. Thus H_1, imbedded into $\mathrm{U}_{\mathbb{K}}(I)$ in this way, is Howe, and (D_1, D_2) is a direct successor of the Howe dual pair $(H_1, C_{\mathrm{U}_{\mathbb{K}}(I)}(H_1))$. By Remark 1, $C_{\mathrm{U}_{\mathbb{K}}(I)}(H_1) = H_2$.

Now turn to the converse assertion. Let $(H_1, H_2) = (\mathrm{GL}(l_1, \mathbb{L}_1), \mathrm{GL}(l_2, \mathbb{L}_2))$ and $(D_1, D_2) = (\mathrm{GL}(m_1, \mathbb{M}_1), \mathrm{GL}(m_2, \mathbb{M}_2))$ be type 2 IRHDP in $\mathrm{U}_{\mathbb{K}}(I)$. Assume that (D_1, D_2) is a direct successor of (H_1, H_2). Then H_1 acts, as a subgroup of D_1, on $\mathbb{M}_1^{m_1}$. By (7), this representation decomposes, over the center $\mathbb{K}'$ of $\mathbb{K}$, into a number of fundamental irreps of H_1:

$$\mathbb{M}_1^{m_1} = (\mathbb{L}_1^{l_1})^a.$$

On the other hand, this representation is irreducible over $\mathbb{M}_1$: otherwise there was an H_1-invariant decomposition $\mathbb{M}_1^{m_1} = X \oplus Y$ over $\mathbb{M}_1$, and $\mathrm{GL}(X) \times \mathrm{GL}(Y)$ would generate, as a subgroup of D_1, a Howe subgroup of $\mathrm{U}_{\mathbb{K}}(I)$ separating H_1 and D_1.

Thus, irreducibility implies:
- If $\mathbb{L}_1 \subseteq \mathbb{M}_1$ then $a = \dim_{\mathbb{L}_1} \mathbb{M}_1$. Hence in this case $m_1 = l_1$.
- If $\mathbb{M}_1 \subseteq \mathbb{L}_1$ then $a = 1$, since irreducibility over $\mathbb{M}_1$ implies irreducibility over $\mathbb{L}_1$. So in this case $m_1 = bl_1$, where $b = \dim_{\mathbb{M}_1} \mathbb{L}_1$.

As an immediate consequence, $\mathbb{L}_1 \neq \mathbb{M}_1$. Moreover, it is obvious that in the field extensions $\mathbb{L}_1 \subset \mathbb{M}_1$ or $\mathbb{M}_1 \subset \mathbb{L}_1$, respectively, the situation $\mathbb{R} \subseteq \mathbb{H}$ cannot occur. So D_1 is contained in Table (32). $\square$

6.3. Type 1 irreducible direct successors of type 1 IRHDP

In this section, for brevity of notation we shall call the passage from $\mathbb{R}$ to $\mathbb{C}_1$ or $\mathbb{C}_c$, and from $\mathbb{C}_c$ to $\mathbb{H}$ a *minimal involutive field extension*.

Proposition 3. *Let* $(H_1, H_2) = (\mathrm{U}_{\mathbb{L}_1}(J_1), \mathrm{U}_{\mathbb{L}_2}(J_2))$ *and* $(D_1, D_2) = (\mathrm{U}_{\mathbb{M}_1}(K_1), \mathrm{U}_{\mathbb{M}_2}(K_2))$ *be IRHDP of type* 1 *in* $\mathrm{U}_{\mathbb{K}}(I)$. *Then* (D_1, D_2) *is a direct successor of* (H_1, H_2) *iff either*
(a) $\mathbb{M}_1$ *is a minimal involutive field extension of* $\mathbb{L}_1$ *and*

$$l_1 = m_1, \quad K_1 = J_1,$$
$$J_2 = \begin{cases} \widehat{K_2}, & \mathbb{K} = \mathbb{R},\ \mathbb{M}_2 \neq \mathbb{C}_1, \\ \Delta_{l_2}\widehat{K_2}, & \mathbb{K} = \mathbb{R},\ \mathbb{M}_2 = \mathbb{C}_1, \\ K_2, & \mathbb{K} = \mathbb{H}, \end{cases}$$

or
(b) $\mathbb{L}_1$ *is a minimal involutive field extension of* $\mathbb{M}_1$ *and*

$$m_1 = 2l_1,$$
$$K_1 = \begin{cases} \widehat{J_1}, & \mathbb{L}_1 \neq \mathbb{C}_1, \\ \Delta_{m_1}\widehat{J_1}, & \mathbb{L}_1 = \mathbb{C}_1, \end{cases} \qquad J_2 = \begin{cases} K_2, & \mathbb{K} = \mathbb{R}, \\ \widehat{K_2}, & \mathbb{K} = \mathbb{H}. \end{cases}$$

Remarks.
1. Depending on the values of $\mathbb{L}_i$ and $\mathbb{M}_i$, $i = 1, 2$, the matrices $\widehat{K_2}$ and $\widehat{J_1}$ are the images of K_2 and J_1 under the imbeddings (12) or (14), respectively.
2. We shall say that (D_1, D_2) arises from (H_1, H_2) by *involutive field extension* (case (a)) or *restriction* (case (b)), respectively.
3. Note that the relations between metrics are understood modulo similarity. So in order to obtain all solutions K_1, K_2 one has to run J_1, J_2 through the respective similarity class, with the constraint that (D_1, D_2) is a Howe dual pair in $\mathrm{U}_{\mathbb{K}}(I)$.
4. Similar to type 2, for $\mathbb{K} = \mathbb{C}$ there are no type 1 irreducible direct successors of type 1 IRHDP.

Proof. To begin with, we shall show that any type 1 direct successor of (H_1, H_2) is subject to either condition (a) or (b). $\square$

Lemma 6. *Assume that* (D_1, D_2) *is a direct successor of* (H_1, H_2). *Then either* $\mathbb{L}_1 \subset \mathbb{M}_1$ *and* $l_1 = m_1$, *or* $\mathbb{M}_1 \subset \mathbb{L}_1$ *and* $m_1 = bl_1$, *where* $b = \dim_{\mathbb{L}_1} \mathbb{M}_1$.

Proof. The proof goes along the lines of the second part of the proof of Proposition 2. At first we shall show that the action of $\mathrm{gl}(l_1, \mathbb{L}_1)$ on $\mathbb{M}_1^{m_1}$, which is induced by the inclusion $H_1 \subset D_1$, is irreducible: Assume that there is a non-trivial subspace $X \subset \mathbb{M}_1^{m_1}$ invariant under $\mathrm{gl}(l_1, \mathbb{L}_1)$. Let $X^\perp$ denote its orthogonal complement in $(\mathbb{M}_1^{m_1}, K_1)$. Consider the Howe subgroup S of $\mathrm{U}_{\mathbb{K}}(I)$ generated by the stabilizer

$$S_0 := \{A \in D_1 : A(X \cap X^\perp) \subseteq (X \cap X^\perp)\}$$

of $X \cap X^\perp$ w.r.t. D_1. By $H_1 \subseteq S \subseteq D_1$, either $S = H_1$ or $S = D_1$. Moreover, by Witt's theorem, prolongation to $\mathbb{M}_1^{m_1}$ yields an imbedding $\mathrm{GL}(X \cap X^-) \subseteq S_0$. Hence $S = D_1$. Then, however, $X \cap X^\perp = 0$, since otherwise there was $A \in D_1$ commuting with S_0 but not with D_1. As a consequence,

$$(\mathbb{M}_1^{m_1}, K_1) = (X, K_1^1) \oplus (X^\perp, K_1^2)$$

for some Hermitian metrics K_1^1, K_1^2 over $\mathbb{M}_1$. Then $\mathrm{U}_{\mathbb{M}_1}(K_1^1) \times \mathrm{U}_{\mathbb{M}_1}(K_1^2)$ generates a Howe subgroup of $\mathrm{U}_{\mathbb{K}}(I)$ separating H_1 and D_1 (contradiction). Thus, $\mathrm{gl}(l_1, \mathbb{L}_1)$ acts irreducibly on $\mathbb{M}_1^{m_1}$.

Now an argumentation similar to the one in the proof of Proposition 2 shows that either $\mathbb{L}_1 \subset \mathbb{M}_1$ and $l_1 = m_1$, or $\mathbb{M}_1 \subset \mathbb{L}_1$ and $m_1 = bl_1$, where $b = \dim_{\mathbb{L}_1} \mathbb{M}_1$.

It remains to check that these field extensions are involutive. In case $\mathbb{K} = \mathbb{H}$ this is obvious. In case $\mathbb{K} = \mathbb{R}$, on the other hand, one may assume $\mathbb{L}_1 \subset \mathbb{M}_1$ and $l_1 = m_1$ (otherwise $\mathbb{M}_2 \subset \mathbb{L}_2$ and $m_2 = l_2$). Then the inclusion is induced by the imbedding $H_1 \subset D_1$ and hence is involutive by (19).

To proceed with the proof of the proposition, we shall derive relations between K_1 and J_1, and between K_2 and J_2 by exploiting the inclusion relations $H_1 \subset D_1$ and $D_2 \subset H_2$, respectively. To this end we shall sort these relations into two classes and apply the following lemma:

Lemma 7. *Let $\mathbb{L}$, $\mathbb{M}$ be involutive fields such that $\mathbb{L} \subset \mathbb{M}$. Let J, K be metrics of dimension l, m over $\mathbb{L}$, $\mathbb{M}$, respectively. Assume that $\mathrm{U}_{\mathbb{L}}(J)$ and $\mathrm{U}_{\mathbb{M}}(K)$ are Howe subgroups of $\mathrm{U}_{\mathbb{K}}(I)$. Consider the following two types of inclusion relations:*

(A) $\mathrm{U}_{\mathbb{L}}(J) \subset \mathrm{U}_{\mathbb{M}}(K)$, *where* $l = m$,

(B) $\mathrm{U}_{\mathbb{M}}(K) \subset \mathrm{U}_{\mathbb{L}}(J)$, *where* $l = m \dim_{\mathbb{L}} \mathbb{M}$.

Assume that the RHS in both cases is a direct successor of the LHS. Then $\mathbb{M}$ is a minimal involutive field extension of $\mathbb{L}$. Moreover, in case (A), *$K = J$, whereas in case* (B),

$$\widehat{K} = \begin{cases} \Delta_{bm} J, & \mathbb{L} = \mathbb{R}, \mathbb{M} = \mathbb{C}_1, \\ J, & \text{otherwise.} \end{cases}$$

Proof. Consider at first case (A). Here $\mathrm{gl}(l, \mathbb{L}) \subset \mathrm{gl}(l, \mathbb{M})$ and $A^J = A^K$ for any $A \in \mathrm{gl}(l, \mathbb{L})$. Then $J^{-1}K$, as an element of $\mathrm{gl}(l, \mathbb{M})$, commutes with $\mathrm{gl}(l, \mathbb{L})$. Hence

$$K = J\alpha \quad \text{for some } \alpha \in C_{\mathbb{M}}(\mathbb{L}). \tag{33}$$

Now consider the possible combinations of $\mathbb{L}$ and $\mathbb{M}$ separately:
1. $\mathbb{L} = \mathbb{R}, \mathbb{M} = \mathbb{C}_1, \mathbb{C}_c$: $\alpha \in \mathbb{C}$, hence $K = J$ up to similarity of both J and K.

2. $\mathbb{L} = \mathbb{C}_c$, $\mathbb{M} = \mathbb{H}$: $\alpha \in \mathbb{C}$, hence again $K = J$, modulo similarity.
3. $\mathbb{L} = \mathbb{R}$, $\mathbb{M} = \mathbb{H}$: $\alpha \in \mathbb{H}$, hence, up to similarity, $K^{(1)} = J$, and $K^{(2)} = \mathrm{i}J$. However, both $K^{(1)}$ and $K^{(2)}$ are also metrics over $\mathbb{C}_c$. So $\mathrm{U}_{\mathbb{C}_c}(K^{(j)})$ generates a Howe subgroup of $\mathrm{U}_{\mathbb{K}}(I)$ separating $\mathrm{U}_{\mathbb{L}}(J)$ and $\mathrm{U}_{\mathbb{M}}(K)$. Thus, case 3 does not occur.

Now turn to type (B). Denote $b := \dim_{\mathbb{L}} \mathbb{M}$. Inclusion (B) yields an imbedding $\mathrm{gl}(m, \mathbb{M}) \subset \mathrm{gl}(bm, \mathbb{L})$ which is equivalent to the standard one $A \mapsto \widehat{A}$, where $\widehat{A}$ is given by (12), (14), or (16), respectively (depending on the values of $\mathbb{L}$ and $\mathbb{M}$). Then with J and K possibly modified up to similarity, $\widehat{A^K} = \widehat{A}^J$ for any $A \in \mathrm{U}_{\mathbb{M}}(K)$. Similar to (A) one obtains, using (23),

$$\widehat{K} = \begin{cases} \Delta_{bm} J\widehat{\alpha}, & \mathbb{L} = \mathbb{R},\ \mathbb{M} = \mathbb{C}_1, \\ J\widehat{\alpha}, & \text{otherwise}, \end{cases} \tag{34}$$

for some $\alpha \in \mathbb{M}'$. Finally, a discussion of the possible combinations of $\mathbb{L}$ and $\mathbb{M}$ yields the assertion.

By Lemma 7 and the following table, which is derived from Lemma 6, one obtains the relations between J_i and K_i, $i = 1, 2$ which are asserted in the proposition.

$\mathbb{K}$	Relation between $\mathbb{L}_1$ and $\mathbb{M}_1$	Type of $H_1 \subset D_1$	Type of $D_2 \subset H_2$
$\mathbb{R}$	$\mathbb{L}_1 \subset \mathbb{M}_1$	(A)	(B)
	$\mathbb{M}_1 \subset \mathbb{L}_1$	(B)	(A)
$\mathbb{H}$	$\mathbb{L}_1 \subset \mathbb{M}_1$	(A)	(A)
	$\mathbb{M}_1 \subset \mathbb{L}_1$	(B)	(B)

For the converse direction of the proposition assume that (D_1, D_2) obeys condition (a) or (b) of the proposition. A standard argument indicated in the proof of Proposition 2 shows that H_1, imbedded into $\mathrm{U}_{\mathbb{K}}(I)$ as a subgroup of some representative of D_1, is Howe, and has direct successor D_1. Moreover, the centralizer $\tilde{H}_2$ of H_1 in $\mathrm{U}_{\mathbb{K}}(I)$ has isomorphism class $\mathrm{U}_{\mathbb{L}_2}(\tilde{J}_2)$ where $\tilde{J}_2$ is subject to condition (18). Since type 1 IRHDP are, in general, not uniquely determined by one of their components it remains to check isomorphy of H_2 and $\tilde{H}_2$, i.e. similarity of $\tilde{J}_2$ and J_2.

Since $(\tilde{H}_2, H_1)$ is a direct successor of (D_2, D_1), the inclusion $D_2 \subset \tilde{H}_2$ belongs to either type (A) or (B). In case (A), by (33), $\tilde{J}_2 = J_2\alpha$, where $\alpha \in C_{\mathbb{M}_2}(\mathbb{L}_2)$. However, both $\tilde{J}_2$ and J_2 have entries in $\mathbb{L}_2$, so that α is also an element of $\mathbb{L}_2$. It follows that $\alpha \in \mathbb{L}_2'$, and $\tilde{J}_2$ and J_2 are similar.

This argument applies if at least one of the inclusion relations $H_1 \subset D_1$ and $D_2 \subset H_2$ is of type (A). If both are of type (B) then $\mathbb{K} = \mathbb{H}$. One may assume that $(\mathbb{L}_1, \mathbb{L}_2) = (\mathbb{C}_c, \mathbb{C}_c)$ and $(\mathbb{M}_1, \mathbb{M}_2) = (\mathbb{R}, \mathbb{H})$ (otherwise one proves, from the beginning, that (H_2, H_1) is a direct successor of (D_2, D_1)). By (34), $\tilde{J}_2 = J_2\widehat{\alpha}$, where $\alpha \in \mathbb{H}' = \mathbb{R}$. Thus $\tilde{J}_2$ and J_2 are similar in this case, too. □

Since Proposition 3 is not very explicit yet, it proves useful to derive a list of type 1 direct successors of type 1 IRHDP from it (see Table 5). The following examples shall give an idea of how this may be done. We shall restrict our attention to $\mathrm{O}(p, p)$.

Table 5
Type 1 irreducible direct successors of type 1 IRHDP in $U_{\mathbb{K}}(l)$

$U_{\mathbb{K}}(l)$	IRHDP	Direct successors	Conditions
$O(p,q)$	$O(p_1,q_1), O(p_2,q_2)$	$U(p_1,q_1), U(\frac{1}{2}p_2,\frac{1}{2}q_2)$	p_2, q_2 even
		$O(p_1+q_1,\mathbb{C}), O(p_2,\mathbb{C})$	$p_1+q_1 \neq 1,\ p_2=q_2\neq 1$
	$Sp(n_1,\mathbb{R}), Sp(n_2,\mathbb{R})$	$U(n_1,n_1), U(p_2,q_2)$	$p_2+q_2=n_2$
		$Sp(n_1,\mathbb{C}), Sp(\frac{1}{2}n_2,\mathbb{C})$	n_2 even
	$O(n_1,\mathbb{C}), O(n_2,\mathbb{C})$	$O(n_1,n_1), O(p_2,q_2)$	$p_2+q_2=n_2$
	$Sp(n_1,\mathbb{C}), Sp(n_2,\mathbb{C})$	$Sp(2n_1,\mathbb{R}), Sp(n_2,\mathbb{R})$	
	$U(p_1,q_1), U(p_2,q_2)$	$Sp(p_1,q_1), Sp(\frac{1}{2}p_2,\frac{1}{2}q_2)$	p_2, q_2 even
		$O^*(p_1+q_1), O^*(p_2)$	$p_1+q_1\neq 1,\ p_2=q_2\neq 1$
		$O(2p_1,2q_1), O(p_2,q_2)$	
		$Sp(p_1+q_1,\mathbb{R}), Sp(p_2,\mathbb{R})$	$p_2=q_2$
	$Sp(p_1,q_1), Sp(p_2,q_2)$	$U(2p_1,2q_1), U(p_2,q_2)$	
	$O^*(n_1), O^*(n_2)$	$U(n_1,n_1), U(p_2,q_2)$	$p_2+q_2=n_2$
$Sp(n,\mathbb{R})$	$O(p_1,q_1), Sp(n_2,\mathbb{R})$	$U(p_1,q_1), U(p_2,q_2)$	$p_2+q_2=n_2$
		$O(p_1+q_1,\mathbb{C}), Sp(\frac{1}{2}n_2,\mathbb{C})$	n_2 even, $p_1+q_1\neq 1$
	$Sp(n_1,\mathbb{R}), O(p_2,q_2)$	$U(n_1,n_1), U(\frac{1}{2}p_2,\frac{1}{2}q_2)$	p_2, q_2 even
		$Sp(n_1,\mathbb{C}), O(p_2,\mathbb{C})$	$p_2=q_2\neq 1$
	$O(n_1,\mathbb{C}), Sp(n_2,\mathbb{C})$	$O(n_1,n_1), Sp(n_2,\mathbb{R})$	
	$Sp(n_1,\mathbb{C}), O(n_2,\mathbb{C})$	$Sp(2n_1,\mathbb{R}), O(p_2,q_2)$	$p_2+q_2=n_2$
	$U(p_1,q_1), U(p_2,q_2)$	$Sp(p_1,q_1), O^*(p_2)$	$p_2=q_2\neq 1$
		$O^*(p_1+q_1), Sp(\frac{1}{2}p_2,\frac{1}{2}q_2)$	$p_1+q_1\neq 1$, p_2, q_2 even
		$O(2p_1,2q_1), Sp(p_2,\mathbb{R})$	$p_2=q_2$
		$Sp(p_1+q_1,\mathbb{R}), O(p_2,q_2)$	
	$Sp(p_1,q_1), O^*(n_2)$	$U(2p_1,2q_1), U(p_2,q_2)$	$p_2+q_2=n_2$
	$O^*(n_1), Sp(p_2,q_2)$	$U(n_1,n_1), U(p_2,q_2)$	$p_1+q_1=n_1$
$Sp(p,q)$	$O(p_1,q_1), Sp(p_2,q_2)$	$U(p_1,q_1), U(p_2,q_2)$	
	$Sp(n_1,\mathbb{R}), O^*(n_2)$	$U(n_1,n_1), U(p_2,q_2)$	$p_2+q_2=n_2$
	$U(p_1,q_1), U(p_2,q_2)$	$Sp(p_1,q_1), O(p_2,q_2)$	
		$O(2p_1,2q_1), Sp(\frac{1}{2}p_2,\frac{1}{2}q_2)$	p_2, q_2 even
		$O^*(p_1+q_1), Sp(p_2,\mathbb{R})$	$p_2=q_2,\ p_1+q_1\neq 1$
		$Sp(p_1+q_1,\mathbb{R}), O^*(p_2)$	$p_2=q_2\neq 1$
	$Sp(p_1,q_1), O(p_2,q_2)$	$U(2p_1,2q_1), U(\frac{1}{2}p_2,\frac{1}{2}q_2)$	p_2, q_2 even
	$O^*(n_1), Sp(n_2,\mathbb{R})$	$U(n_1,n_1), U(p_2,q_2)$	$p_2+q_2=n_2$
$O^*(n)$	$O(p_1,q_1), O^*(n_2)$	$U(p_1,q_1), U(p_2,q_2)$	$p_2+q_2=n_2$
$(n\neq 1)$	$Sp(n_1,\mathbb{R}), Sp(p_2,q_2)$	$U(n_1,n_1), U(p_2,q_2)$	
	$U(p_1,q_1), U(p_2,q_2)$	$O^*(p_1+q_1), O(p_2,q_2)$	
		$O(2p_1,2q_1), O^*(p_2)$	$p_2=q_2\neq 1$
		$Sp(p_1,q_1), Sp(p_2,\mathbb{R})$	$p_2=q_2$
		$Sp(p_1+q_1,\mathbb{R}), Sp(\frac{1}{2}p_2,\frac{1}{2}q_2)$	p_2, q_2 even
	$O^*(n_1), O(p_2,q_2)$	$U(n_1,n_1), U(\frac{1}{2}p_2,\frac{1}{2}q_2)$	p_2, q_2 even
	$Sp(p_1,q_1), Sp(n_2,\mathbb{R})$	$U(2p_1,2q_1), U(p_2,q_2)$	$p_2+q_2=n_2$

[a] NOTE. The conditions on the pairs in the 2nd and 3rd column to appear as IRHDP in $U_{\mathbb{K}}(l)$ have already been displayed in Table 4, and henceforth are omitted here.

Example 5. To begin with, let us derive the type 1 irreducible direct successors of the pair $(O(p_1,q_1), O(p_2,q_2))$. Here $\mathbb{L}_1=\mathbb{R}$ so that only case (a) can occur. As a consequence, $K_1=J_1$.

Consider at first the involutive field extension $\mathbb{M}_1 = \mathbb{C}_1$. Since K_1 has flip factor $\varepsilon = 1$, $D_1 = \mathrm{O}(p_1 + q_1, \mathbb{C})$. The relation between K_2 and J_2 is $\widehat{K_2} = \Delta_{p_2+q_2} J_2$. It can only have a solution if $p_2 = q_2$. In this case one may choose $J_2 = \Delta_{p_2+q_2}$ to obtain $K_2 = \mathbf{1}_{p_2}$ and $D_2 = \mathrm{O}(p_2, \mathbb{C})$. So the involutive field extension $\mathbb{M}_1 = \mathbb{C}_1$ yields the direct successor $(\mathrm{O}(p_1+q_1, \mathbb{C}), \mathrm{O}(p_2, \mathbb{C}))$ of the pair $(\mathrm{O}(p_1, q_1), \mathrm{O}(p_2, p_2))$. Furthermore, as an immediate consequence,

$$(\mathrm{O}(n_1, n_1), \mathrm{O}(p_2, n_2 - p_2)), \quad \tfrac{1}{2}n_2 \le p_2 \le n_2,$$

are direct successors of the pair $(\mathrm{O}(n_1, \mathbb{C}), \mathrm{O}(n_2, \mathbb{C}))$, which are obtained by involutive field restriction. (Clearly, in case $\mathbb{K} = \mathbb{R}$ it suffices to determine the type 1 direct successors obtained by field *extension*.)

Next consider the involutive field extension $\mathbb{M}_1 = \mathbb{C}_c$. Here $D_1 = \mathrm{U}(p_1, q_1)$. The equation $\widehat{K_2} = J_2$ requires p_2, q_2 to be even. In this case one may put $J_2 = \mathrm{diag}(\mathbf{1}_{p_2}, -\mathbf{1}_{q_2})$, thus obtaining $D_2 = \mathrm{U}(\frac{1}{2}p_2, \frac{1}{2}q_2)$. Again, the field extension $\mathbb{M}_1 = \mathbb{C}_c$ also gives rise to the direct successor $(\mathrm{O}(2p_1, 2q_1), \mathrm{O}(p_2, q_2))$ of the pair $(\mathrm{U}(p_1, q_1), \mathrm{U}(p_2, q_2))$, which is actually obtained by field restriction.

Example 6. Now turn to the pair $(\mathrm{Sp}(n_1, \mathbb{R}), \mathrm{Sp}(n_2, \mathbb{R}))$. For the involutive field extension $\mathbb{M}_1 = \mathbb{C}_1$ one finds that n_2 must be even, and $(D_1, D_2) = (\mathrm{Sp}(n_1, \mathbb{C}), \mathrm{Sp}(\frac{1}{2}n_2, \mathbb{C}))$. For the involutive field extension $\mathbb{M}_1 = \mathbb{C}_c$, on the other hand, one may choose for J_1 the usual symplectic matrix. Then J_1 has eigenvalues i and $-$i, each one with multiplicity n_1. Thus $D_1 = \mathrm{U}(n_1, n_1)$. Moreover, given a decomposition $n_2 = p_2 + q_2$, put $K_2 = \mathrm{diag}(\mathrm{i}\mathbf{1}_{p_2}, -\mathrm{i}\mathbf{1}_{q_2})$. Then $\widehat{K_2}$ is a real symplectic metric, hence may serve as J_2. Thus $D_2 = \mathrm{U}(p_2, q_2)$, where $p_2 + q_2 = n_2$. (Note that here it proves to be necessary to have J_2 run through its similarity class.)

6.4. Type 2 irreducible direct successors of type 1 IRHDP

Proposition 4. *Assume that I is hyperbolic.*

(a) *Let $(H_1, H_2) = (\mathrm{U}_{\mathbb{L}_1}(J_1), \mathrm{U}_{\mathbb{L}_2}(J_2))$ be a type 1 IRHDP of $\mathrm{U}_{\mathbb{K}}(I)$. Then (H_1, H_2) possesses a type 2 irreducible direct successor (D_1, D_2) iff J_2 is hyperbolic. In this case,*

$$(D_1, D_2) = (\mathrm{GL}(l_1, \mathbb{L}_1), \mathrm{GL}(\tfrac{1}{2}l_2, \mathbb{L}_2)),$$

where l_1 and l_2 denote the dimension of J_1 and J_2, respectively.

(b) *Let $(H_1, H_2) = (\mathrm{GL}(l_1, \mathbb{L}_1), \mathrm{GL}(l_2, \mathbb{L}_2))$ be a type 2 IRHDP of $\mathrm{U}_{\mathbb{K}}(I)$. Then the type 1 irreducible direct successors of (H_1, H_2) are*

$$(\mathrm{U}_{\mathbb{L}_1}(J_1), \mathrm{U}_{\mathbb{L}_2}(J_2)),$$

where J_1 is hyperbolic of dimension $2l_1$.

Proof. Obviously, assertion (b) is dual to (a) by taking the centralizer in $\mathrm{U}_{\mathbb{K}}(I)$. So one only has to prove (a).

To begin with, assume that $(D_1, D_2) = (\mathrm{GL}(m_1, \mathbb{M}_1), \mathrm{GL}(m_2, \mathbb{M}_2))$ is a type 2 direct successor of (H_1, H_2). Then the unitary group $H_2 = \mathrm{U}_{\mathbb{L}_1}(J_2)$ contains D_2 as a non-trivial general linear subgroup. As a consequence, the Hermitian space $(\mathbb{L}_2^{l_2}, J_2)$ contains a non-zero isotropic $\mathbb{L}_2$-subspace X such that $D_2 \subseteq \mathrm{GL}(X) \subseteq H_2$. Since there is non-central $A \in H_2$ commuting with $\mathrm{GL}(X)$, the Howe subgroup of $\mathrm{U}_{\mathbb{K}}(I)$ generated by $\mathrm{GL}(X)$ (and hence $\mathrm{GL}(X)$ itself) coincides with D_2. This implies $\mathbb{L}_2 = \mathbb{M}_2$ and $X \cong \mathbb{L}_2^{m_2}$. In particular, $\mathbb{L}_1 = \mathbb{M}_1$. It follows $l_1 = m_1$, because otherwise $\mathrm{GL}(l_1, \mathbb{L}_1)$ would generate a Howe subgroup of $\mathrm{U}_{\mathbb{K}}(I)$ separating H_1 and D_1. By $l_1 l_2 = 2m_1 m_2$, $l_2 = 2m_2$. Thus, X is an isotropic subspace of half dimension of $(\mathbb{L}_2^{l_2}, J_2)$. As a consequence, J_2 is hyperbolic.

Conversely, assume that J_2 is hyperbolic. Then l_2 is even and $\mathrm{GL}(\frac{1}{2}l_2, \mathbb{L}_2) \subset H_2$. Since H_2 is subject to condition (19), there is no general linear nor unitary group nor a product thereof separating $\mathrm{GL}(\frac{1}{2}l_2, \mathbb{L}_2)$ and H_2. Hence $\mathrm{GL}(\frac{1}{2}l_2, \mathbb{L}_2)$, imbedded into $\mathrm{U}_{\mathbb{K}}(I)$ in this way, is Howe and generates the Howe dual pair $(D_1, D_2) = (\mathrm{GL}(l_1, \mathbb{L}_1), \mathrm{GL}(\frac{1}{2}l_2, \mathbb{L}_2))$. Moreover, (D_1, D_2) is a direct successor of (H_1, H_2). □

This concludes the discussion of the natural partial ordering relation of Howe dual pairs. In the next section we shall consider the set of reductive Howe dual pairs $\mathcal{H}(G)$ of a few standard groups G in some detail.

7. Examples

In the following, we shall use the direct successor relations established in Section 6 to draw, beginning with the center, Hasse diagrams of $\mathcal{H}(\mathrm{U}_{\mathbb{K}}(I))$. In these diagrams, in order to avoid arrows, we shall adopt the convention where the left vertex of a line is always less than the right one. Moreover, vertices are labeled by the first constituents of the corresponding Howe dual pairs only. The other constituent can be obtained by reflection at the vertical middle axis (this operation corresponds to taking the centralizer in $\mathrm{U}_{\mathbb{K}}(I)$).

Example 7. At first, we shall discuss $\mathrm{U}(n)$, which is the most simple example. The IRHDP are $(\mathrm{U}(n_1), \mathrm{U}(n_2))$, where $n_1 n_2 = n$ (all of type 1). Since $\mathrm{U}(n)$ is defined by a scalar product on $\mathbb{C}_c^n$, any subspace is Hermitian. So Hermitian decompositions of $\mathbb{C}_c^n$ are given by sum decompositions $n = n^1 + \cdots + n^r$. Hence Howe dual pairs are

$$(\mathrm{U}(n_1^1) \times \cdots \times \mathrm{U}(n_1^r), \mathrm{U}(n_2^1) \times \cdots \times \mathrm{U}(n_2^r)), \quad \text{where } \sum_{i=1}^{r} n_1^i n_2^i = n.$$

Direct successors arise solely by splitting and inverse splitting (Proposition 1). For the factors this yields the following two generating direct successor relations:

$$\begin{aligned}&(\mathrm{U}(n_1^i), \mathrm{U}(n_2^i)) \leq (\mathrm{U}(n_1^i) \times \mathrm{U}(n_1^i), \mathrm{U}(l_2^i) \times \mathrm{U}(m_2^i)), \quad \text{where } l_2^i + m_2^i = n_2^i,\\ &(\mathrm{U}(l_1^i) \times \mathrm{U}(m_1^i), \mathrm{U}(n_2^i) \times \mathrm{U}(n_2^i)) \leq (\mathrm{U}(l_1^i + m_1^i), \mathrm{U}(n_2^i)).\end{aligned}$$

As an example, we draw the Hasse diagrams of $\mathcal{H}(\mathrm{U}(2))$ and $\mathcal{H}(\mathrm{U}(3))$ in Fig. 1 as well as the one of $\mathcal{H}(\mathrm{U}(5))$ in Fig. 2.

Fig. 1. Hasse diagrams of $\mathcal{H}(\mathrm{U}(2))$ (left) and $\mathcal{H}(\mathrm{U}(3))$ (right).

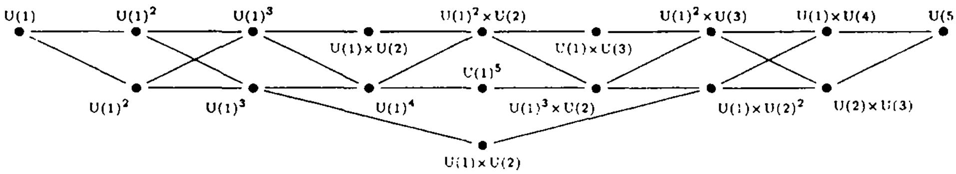


Fig. 2. Hasse diagram of $\mathcal{H}(\mathrm{U}(5))$.

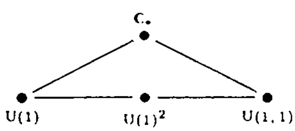


Fig. 3. Hasse diagram of $\mathcal{H}(\mathrm{U}(1,1))$.

Remarks.
1. At least $\mathcal{H}(\mathrm{U}(2))$ and $\mathcal{H}(\mathrm{U}(3))$ are well known. The first one, for instance, has been used in [2], and the second one in [4].
2. The sets of Howe dual pairs of $\mathrm{U}(n)$ and $\mathrm{GL}(n, \mathbb{C})$ are isomorphic. In general, if G is a complex Lie group and H its compact real form then the Howe dual pairs of H are the compact real forms of the Howe dual pairs of G.

Example 8. Next consider $\mathrm{U}(1, 1)$. Since the corresponding metric is hyperbolic, there is, besides the trivial IRHDP, a type 2 IRHDP, namely $(\mathrm{GL}(1, \mathbb{C}), \mathrm{GL}(1, \mathbb{C}))$. Moreover, there is a single Hermitian decomposition of the metric: $\mathrm{diag}(1, -1) = (1) \oplus (1)$. Let us draw the Hasse diagram. The center $(\mathrm{U}(1), \mathrm{U}(1, 1))$ has direct successors $(\mathrm{U}(1), \mathrm{U}(1))^2$ (obtained by splitting), and $(\mathrm{GL}(1, \mathbb{C}), \mathrm{GL}(1, \mathbb{C}))$ (by virtue of Proposition 4). Both $(\mathrm{U}(1), \mathrm{U}(1)) \times (\mathrm{U}(1), \mathrm{U}(1))$ and $(\mathrm{GL}(1, \mathbb{C}), \mathrm{GL}(1, \mathbb{C}))$ then have direct successor $(\mathrm{U}(1, 1), \mathrm{U}(1))$. Thus, using the notation $\mathbb{K}_* := \mathrm{GL}(1, \mathbb{K})$, the Hasse diagram is as shown in Fig. 3.

Note that $\mathbb{C}_*$, if viewed as subgroup of the real Lie group $\mathrm{U}(1, 1)$, is in fact the realification of the underlying complex group. So when complexifying again one obtains $\mathbb{C}_*^2$. This shows that the reductive Howe dual pair $(\mathbb{C}_*, \mathbb{C}_*)$ in $\mathrm{U}(1, 1)$ is a real form of the reductive Howe dual pair $(\mathbb{C}_*^2, \mathbb{C}_*^2)$ in the complexification $(\mathrm{GL}(2, \mathbb{C}), \mathrm{GL}(2, \mathbb{C}))$. The other real form of this pair which is contained in $\mathcal{H}(\mathrm{U}(1, 1))$ is $(\mathrm{U}(1)^2, \mathrm{U}(1)^2)$.

Fig. 4 the reductive Howe dual pairs of $\mathrm{U}(1, 2)$, derived in a similar way:

Here there are two Howe subgroups of isomorphism class $\mathrm{U}(1)^2$. Thus, we see that a reductive Howe dual pair (H_1, H_2) in a complex group G may split into several reductive Howe dual pairs in a real form of G not only because of the different real forms of (H_1, H_2) but also because isomorphic real forms of different representatives of (H_1, H_2) may not be conjugate in the real form of G.

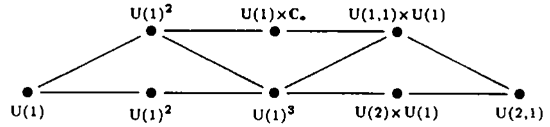


Fig. 4. Hasse diagram of $\mathcal{H}(\mathrm{U}(2, 1))$.

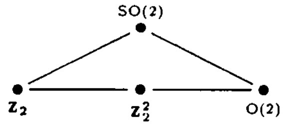


Fig. 5. Hasse diagram of $\mathcal{H}(\mathrm{O}(2))$.

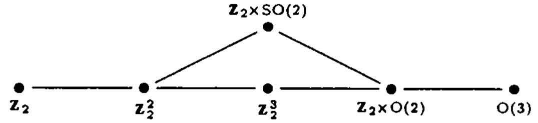


Fig. 6. Hasse diagram of $\mathcal{H}(\mathrm{O}(3))$.

Example 9. Now let us turn to the case of real orthogonal groups $\mathrm{O}(n)$. Here the IRHDP are: $(\mathrm{O}(n_1), \mathrm{O}(n_2))$, where $n_1 n_2 = n$, $(\mathrm{U}(n_1), \mathrm{U}(n_2))$, where $2n_1 n_2 = n$, and $(\mathrm{Sp}(n_1), \mathrm{Sp}(n_2))$, where $4n_1 n_2 = n$ (all type 1). Similar to Example 1, Hermitian decompositions are given by sum decompositions $n = n^1 + \cdots + n^r$. So direct successors are obtained by splitting and its inverse, as well as involutive field extension and restriction. For O(2), for instance, one finds that the center $(\mathrm{O}(1), \mathrm{O}(2))$ has direct successor $(\mathrm{O}(1), \mathrm{O}(1))^2$ (by splitting) and $(\mathrm{U}(1), \mathrm{U}(1))$ (by involutive field extension). Moreover, both inverse splitting of the first pair and field restriction of the second one yield the direct successor $(\mathrm{O}(2), \mathrm{O}(1))$. Hence if we write $\mathbb{Z}_2$ instead of O(1) and SO(2) instead of U(1) then the Hasse diagram is as shown in Fig. 5.

Fig. 6 shows the Howe dual pairs of O(3). Here the non-trivial Howe subgroups have the following meaning:
- $\mathbb{Z}_2^2$: Reflection at a plane and reflection therein, commuting with
- $\mathbb{Z}_2 \times \mathrm{O}(2)$: Reflection at a plane and O(2) therein,
- $\mathbb{Z}_2 \times \mathrm{SO}(2)$: Reflection at a plane and rotations therein (commuting with itself),
- $\mathbb{Z}_2^3$: Reflections at three independent planes (commuting with itself, too).

Example 10. Consider, as a slightly more challenging example, Lorentz group O(3, 1) (see Fig. 7). (Due to the lack of space brackets are omitted here.)

Example 11. Finally, let us consider $\mathrm{Sp}(2, \mathbb{R})$, as a simple example of a symplectic group. As we have stated in Section 1, the reductive Howe dual pairs of symplectic groups are the ones relevant in representation theory, hence they are very well known. Now, here is their partial ordering (see Fig. 8.)

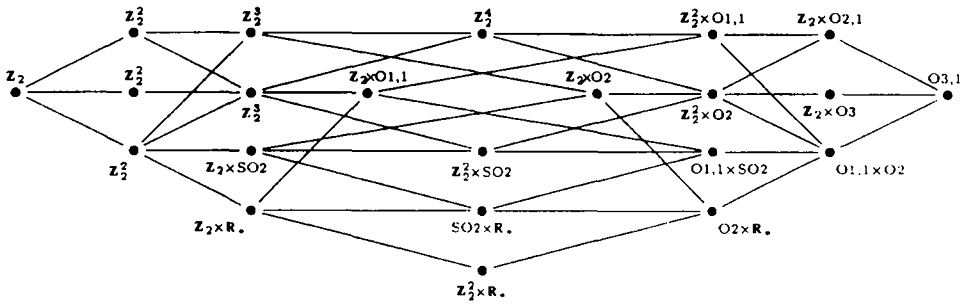


Fig. 7. Hasse diagram of $\mathcal{H}(\mathrm{O}(3,1))$.

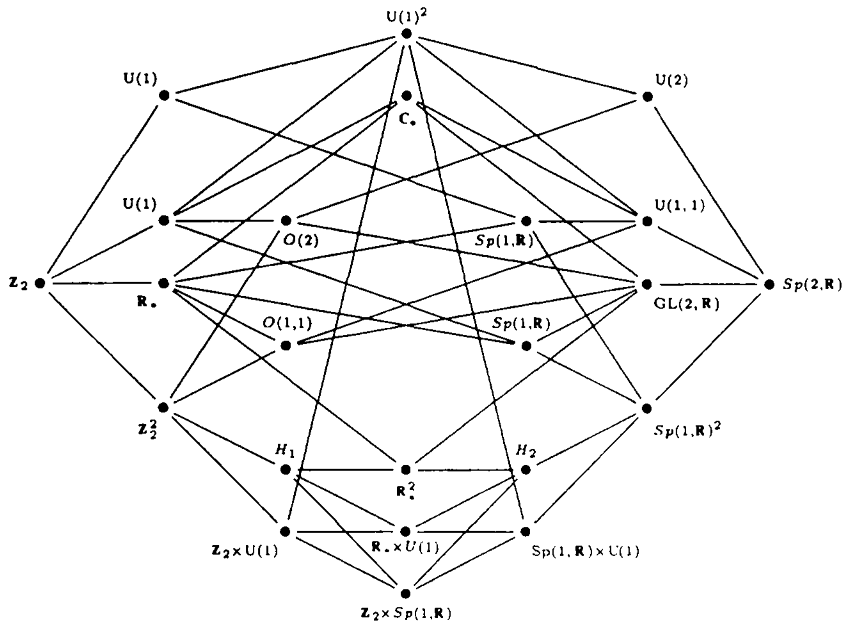


Fig. 8. Hasse diagram of $\mathcal{H}(\mathrm{Sp}(2,\mathbb{R}))$ (here $H_1 = \mathbb{Z}_2 \times \mathbb{R}_*$, $H_2 = \mathrm{Sp}(1,\mathbb{R}) \times \mathbb{R}_*$).

The last two examples illustrate, by the way, that the number of Howe dual pairs rapidly increases with increasing rank. For classical groups of higher rank it will be reasonable to use computer algebra to derive the natural partial ordering relation from direct successor relations.

8. A remark on seesaw pairs

Knowledge of $\mathcal{H}(G)$ yields a solution (not very elegant, though) to the classification problem of so-called *seesaw* pairs [10]. These are pairs of reductive Howe dual pairs (H_1, H_2), (D_1, D_2) in G with the property $H_1 \subset D_1$. Clearly, the listing of these pairs, which we do not carry out here, amounts to an inspection of $\mathcal{H}(G)$.

The notion of a seesaw pair has been introduced by Kudla [10] in connection with considerations about a unified view on identities between inner products of automorphic forms on different groups. In [10], the author gave some examples of seesaw pairs in $\mathrm{Sp}(n, \mathbb{K})$ and expressed the wish to have a classification result. To our knowledge, however, such a result has not been published yet. Now, in view of the direct successor relations derived in Section 6 one can state that the examples given in [10], Section 2, cover all possible direct successor relations in $\mathcal{H}(\mathrm{Sp}(, \mathbb{K}))$. Thus, iterated application of these examples generates all seesaw pairs in $\mathrm{Sp}(n, \mathbb{K})$.

9. An application to Yang–Mills theory

As an application of the theory of RHDP, consider a pure gauge theory with compact internal symmetry G, defined on a principal bundle P over compact space–time X. As outlined in Section 1 we are interested in the singularity structure of the space of gauge orbits $\mathcal{M}$ (connections in P modulo gauge transformations). It is well known [9] that $\mathcal{M}$ is homeomorphic to the orbit space of a differentiable G-action on the manifold of connections modulo pointed gauge transformations. Thus, one has the following facts which are standard for compact group actions [1]: There is a decomposition

$$\mathcal{M} = \bigcup_{\sigma \in \Sigma_P} \mathcal{M}_\sigma, \tag{35}$$

where Σ_P denotes the set of orbit types of this action, and $\mathcal{M}_\sigma$ is the subset of $\mathcal{M}$ consisting of orbits of type σ. Usually, the decomposition (35) is called a stratification, with strata $\mathcal{M}_\sigma$. For any $\sigma \in \Sigma_P$, $\mathcal{M}_\sigma$ is a smooth manifold. Σ_P carries a natural partial ordering which is defined by inclusion modulo conjugacy (recall that the elements of Σ_P are conjugacy classes of subgroups of G). For any $\sigma \in \Sigma_P$, $\mathcal{M}_\sigma$ is open and dense in the union

$$\bigcup_{\sigma' \geq \sigma} \mathcal{M}_{\sigma'}.$$

So one may view the strata $\mathcal{M}_{\sigma'}$, $\sigma' > \sigma$, as singularities in the union. Moreover, the information about which strata occur and how they are patched together is encoded in the partially ordered set Σ_P. Let us refer to Σ_P as the set of orbit types associated to the principal bundle P.

In the following, assume that space–time is homeomorphic to the sphere S^4. From a general classification result [4] it follows that, in this case, Σ_P is the subset of $\mathcal{H}(G)$ consisting of those RHDP (H_1, H_2) for which
(a) H_2 has the same centralizer in G as its 1-component, and
(b) P may be reduced to H_2.
These conditions are due to the fact that any stabilizer subgroup of the action we are considering are given as centralizer, in G, of the holonomy group of some connection in P.

Now specify $G = \mathrm{SU}(n)$. In order to derive $\mathcal{H}(\mathrm{SU}(n))$ from $\mathcal{H}(\mathrm{U}(n))$ we apply the following simple rule: Let $G \subseteq K$ be a subgroup. Then the RHDP of G are

$$(G \cap H_1, G \cap H_2),$$

where (H_1, H_2) runs through all RHDP of K which satisfy

$$G \cap C_K(H_i) = G \cap C_K(G \cap H_i), \quad i = 1, 2.$$

Using the notation $SH_i := \mathrm{SU}(n) \cap H_i$ we find $\mathcal{H}(\mathrm{SU}(n)) = \mathcal{H}(\mathrm{U}(n))$, with elements (SH_1, SH_2) instead of (H_1, H_2). One checks that all these RHDP obey condition (a) (although the subgroups SH_2 are not necessarily connected).

Next let us discuss condition (b). In general, principal bundles over S^4, with structure group a compact Lie group G, are determined by homotopy classes of transition functions $S^3 \to G$, and hence are classified by the elements of the homotopy group $\pi_3(G)$. In particular, since $\pi_3(\mathrm{U}(n)) = \pi_3(\mathrm{SU}(n)) = \mathbb{Z}$ (provided $n \geq 2$), principal bundles with structure group $\mathrm{U}(n)$ or $\mathrm{SU}(n)$ are classified by an integer k (which coincides with the instanton number). Clearly, a G-bundle of class $\alpha \in \pi_3(G)$ is reducible to a subgroup $j : H \to G$ iff there is a transition function $f : S^3 \to G$ of class α and a transition function $g : S^3 \to H$ such that $f = j \circ g$. Thus, the bundle is reducible to H iff α is contained in the image of the induced homomorphism $j_* : \pi_3(H) \to \pi_3(G)$.

We shall calculate $\pi_3(SH_2)$ and the corresponding homomorphism $j_* : \pi_3(SH_2) \to \mathbb{Z}$ for the RHDP of $\mathrm{SU}(n)$. Assume

$$H_i = \mathrm{U}(n_i^1) \times \cdots \times \mathrm{U}(n_i^r), \quad i = 1, 2,$$

where $\sum_{j=1}^r n_1^j n_2^j = n$. Then $j : H_2 \to \mathrm{U}(n)$ maps $(A^1, \ldots, A^r) \in H_2$ on a block diagonal matrix, with blocks $\mathrm{diag}(A^j, \ldots, A^j)$ (n_1^j entries), $j = 1, \ldots, r$. Moreover,

$$SH_2 = \{(A^1, \ldots, A^r) \in H_2 : \det j(A^1, \ldots, A^r) = 1\}.$$

Consider the Lie group homomorphism

$$\mathrm{U}(1) \times SH_2 \to H_2, \quad (\mathrm{e}^{\mathrm{i}\varphi}, A) \mapsto \mathrm{e}^{\mathrm{i}\varphi} A.$$

This homomorphism is surjective and has discrete kernel N. Hence the exact homotopy sequence of the fibration $N \to \mathrm{U}(1) \times SH_2 \to H_2$ yields

$$\pi_3(SH_2) = \pi_3(H_2).$$

As a consequence, over space–time S^4, the set of orbit types associated to an $\mathrm{SU}(n)$-bundle of class $k \in \mathbb{Z}$ coincides with the one associated to a $\mathrm{U}(n)$-bundle of the same class. Denote this set by Σ_k^n.

It is easily seen that the homomorphism induced by $j : H_2 \to \mathrm{U}(n)$ is

$$j_* : \pi_3(H_2) \to \mathbb{Z}, \quad (k_1, \ldots, k_r) \mapsto \sum_{j=1}^r n_1^j k_j. \tag{36}$$

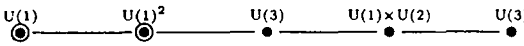
(Here k_j is zero if $n_2^j = 1$ and an arbitrary integer otherwise.) Let $g(H_1, H_2)$ denote the greatest common divisor of those numbers n_1^j, $j = 1, \ldots, r$, for which $n_2^j \neq 1$. Put $g(H_1, H_2) = 0$ if $n_2^j = 1$ for all j. Then (36) yields

$$\operatorname{im} j_* = g(H_1, H_2) \cdot \mathbb{Z}.$$

Thus, we have the following result:

$$\Sigma_k^n = \{(H_1, H_2) \in \mathcal{H}(\mathrm{U}(n)) : g(H_1, H_2) \text{ divides } k\}. \tag{37}$$

As an example, let us consider Σ_k^3. In the Hasse diagram of $\mathcal{H}(\mathrm{U}(3))$ we indicate $g(H_1, H_2)$ by the number of circles surrounding the vertex of (H_1, H_2):



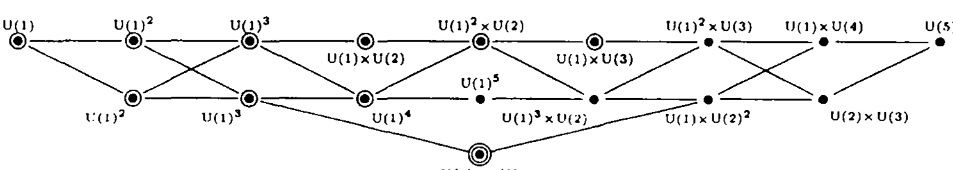
(38)

Thus, the Hasse diagram of Σ_k^3 consists of all vertices in case $k = 0$ (recall that all integers divide 0), and of the vertices surrounded by a circle in case $k \neq 0$, respectively. Next replace (H_1, H_2) by (SH_1, SH_2) in (38). Explicitly, replace $(\mathrm{U}(1), \mathrm{U}(3))$ by $(\mathbb{Z}_3, \mathrm{SU}(3))$, $(\mathrm{U}(1)^2, \mathrm{U}(1) \times \mathrm{U}(2))$ by $(\mathrm{U}(1), \mathrm{U}(2))$, imbedded as

$$\{\operatorname{diag}(\alpha^{-2}, \alpha, \alpha) : \alpha \in \mathrm{U}(1)\} \quad \text{and} \quad \left\{ \begin{pmatrix} (\det A)^{-1} & 0 \\ 0 & A \end{pmatrix} : A \in \mathrm{U}(2) \right\},$$

respectively, and $(\mathrm{U}(1)^3, \mathrm{U}(1)^3)$ by $(\mathrm{U}(1)^2, \mathrm{U}(1)^2)$ (the maximal torus), as well as the first two pairs in the opposite order. Now we can interpret Σ_k^3 as the set of strata of the orbit space of a pure gauge theory defined on a SU(3)-bundle of class k over S^4: The orbit type $(\mathbb{Z}_3, \mathrm{U}(3))$ corresponds to the generic stratum. If the bundle is trivial then there are four additional strata, building up singularities of consecutively increasing degree. When passing to non-trivial bundles, though, there survives only the lowest non-generic stratum.

Analogously, for $\mathcal{H}(\mathrm{U}(5))$ we find



So the Hasse diagram of Σ_k^5 consists either of all vertices (if $k = 0$), of the vertices surrounded by one circle (if k is odd), or of the vertices surrounded by one or two circles (if $k \neq 0$, even). Again, by replacing (H_1, H_2) by (SH_1, SH_2) one obtains the corresponding orbit types for SU(5). Note that in case $k \neq 0$, even, there are two maximal orbit types (or two maximally singular strata, if interpreted as such). By now, we do not know the physical significance of this fact. (Features like that we are going to study in the future.)

Finally, let us discuss which values $g(H_1, H_2)$ may take in $\mathcal{H}(\mathrm{U}(n))$. Clearly,

$$2g(H_1, H_2) \leq n.$$

Conversely, if there is given a positive integer m obeying $2m \leq n$ then

$$(H_1, H_2) = (\mathrm{U}(1)^l \times \mathrm{U}(m), \mathrm{U}(1)^l \times \mathrm{U}(2)), \quad l = n - 2m,$$

is an RDHP in $\mathrm{U}(n)$, and $m = g(H_1, H_2)$. Thus,

$$g(H_1, H_2) = 1, 2, \ldots, [\tfrac{1}{2}n].$$

(Here $[\cdot]$ denotes the integer part.) Hence $\{\Sigma_k^n : k \in \mathbb{Z}\}$ splits into isomorphism classes labeled by those positive integers k which are a least common multiple of some subset of $\{1, 2, \ldots, [\frac{1}{2}n]\}$.

To conclude, we remark that the case of $\mathrm{U}(n)$ (or $\mathrm{SU}(n)$) bundles over space–time S^4 is the simplest one. As a rule, Σ_P will be more interesting for other classical Lie groups. Moreover, Σ_P becomes more sensitive to the topology of P when passing to more complicated space–times. In particular, the sets of orbit types associated to $\mathrm{U}(n)$ and $\mathrm{SU}(n)$-bundles may not coincide any more. Since this subject we are still working on, precise results will be published later.

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